

# DEGENERATE ELLIPTIC OPERATORS IN MATHEMATICAL FINANCE AND HÖLDER CONTINUITY FOR SOLUTIONS TO VARIATIONAL EQUATIONS AND INEQUALITIES

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**ABSTRACT.** The Heston stochastic volatility process, which is widely used as an asset price model in mathematical finance, is a paradigm for a degenerate diffusion process where the degeneracy in the diffusion coefficient is proportional to the square root of the distance to the boundary of the half-plane. The generator of this process with killing, called the elliptic Heston operator, is a second-order degenerate elliptic partial differential operator whose coefficients have linear growth in the spatial variables and where the degeneracy in the operator symbol is proportional to the distance to the boundary of the half-plane. With the aid of weighted Sobolev spaces, we prove supremum bounds, a Harnack inequality, and Hölder continuity near the boundary for solutions to elliptic variational equations defined by the Heston partial differential operator, as well as Hölder continuity up to the boundary for solutions to elliptic variational inequalities defined by the Heston operator. In mathematical finance, solutions to obstacle problems for the elliptic Heston operator correspond to value functions for perpetual American-style options on the underlying asset.

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## 1. INTRODUCTION

Suppose  $\mathcal{O} \subset \mathbb{H}$  is a possibly unbounded domain in the open upper half-plane  $\mathbb{H} := \mathbb{R}^{n-1} \times (0, \infty)$  (where  $n \geq 2$ ),  $\Gamma_1 = \partial\mathcal{O} \cap \mathbb{H}$  is the portion of the boundary  $\partial\mathcal{O}$  of  $\mathcal{O}$  which lies in  $\mathbb{H}$ , and  $\Gamma_0$  is the (non-empty) interior of  $\partial\mathbb{H} \cap \partial\mathcal{O}$ , where  $\partial\mathbb{H} = \mathbb{R}^{n-1} \times \{0\}$  is the boundary of  $\mathbb{H} := \mathbb{R}^{n-1} \times [0, \infty)$ . We require  $\Gamma_0$  to be non-empty and consider a second-order, linear elliptic differential operator,  $A$ , on  $\mathcal{O}$  which is degenerate along  $\Gamma_0$ . Suppose  $f : \mathcal{O} \rightarrow \mathbb{R}$  is a source function. In this article, we prove local supremum bounds near the boundary portion,  $\bar{\Gamma}_0$ , and Hölder continuity up to  $\bar{\Gamma}_0$ , for suitably defined *weak* solutions,  $u : \mathcal{O} \rightarrow \mathbb{R}$ , to the elliptic boundary value problem,

$$Au = f \text{ a.e. on } \mathcal{O}, \quad u = 0 \text{ on } \Gamma_1, \quad (1.1)$$

together with a boundary Harnack inequality (near  $\Gamma_0$ ) for non-negative, *weak* solutions to (1.1) when  $f = 0$ . Because  $A$  is degenerate along  $\Gamma_0$  and weighted Sobolev spaces are required to establish existence of weak solutions to (1.1), these results do not follow from the standard theory for non-degenerate elliptic differential operators [28, 36].

No boundary condition is prescribed in problem (1.1) along  $\Gamma_0$ . Indeed, we recall from [15] that the problem (1.1) is well-posed when we seek solutions in suitable function spaces which describe their qualitative behavior near the boundary portion  $\Gamma_0$ : for example, continuity of derivatives up to  $\Gamma_0$  via suitable weighted Hölder spaces (by analogy with [16]) or integrability of derivatives in a neighborhood of  $\Gamma_0$  via suitable weighted Sobolev spaces (by analogy with [35]).

Similar results were obtained by Koch in the parabolic case [35, Proposition 4.5.1, Theorems 4.5.3 & 4.5.5]. While he used potential theory to obtain the Hölder continuity of solutions and the Harnack inequality, our method of proof is based on the Moser iteration technique. This is not a straightforward adaptation of results [28, Theorems 8.15, 8.20, 8.22 & 8.27], due to

the fact that our Sobolev spaces are weighted, so the standard Sobolev inequality, Poincaré inequality and the John-Nirenberg inequality do not apply. The most difficult step in making the Moser iteration technique work involves a suitable application of the John-Nirenberg inequality. For this purpose, we use the so-called abstract John-Nirenberg inequality, due to Bombieri and Giusti [5, Theorem 4], which can be applied to any topological spaces endowed with a regular Borel measure satisfying some natural requirements. In order to verify the hypotheses of the abstract John-Nirenberg inequality, we prove a local version of the Poincaré inequality, Corollary 2.6, suitable for our weighted spaces.

We also prove Hölder continuity up to  $\bar{\Gamma}_0$  for suitably defined *weak* solutions,  $u : \mathcal{O} \rightarrow \mathbb{R}$ , to the elliptic obstacle problem

$$\min\{Au - f, u - \psi\} = 0 \text{ a.e on } \mathcal{O}, \quad u = 0 \text{ on } \Gamma_1, \quad (1.2)$$

where  $\psi : \mathcal{O} \cup \Gamma_1 \rightarrow \mathbb{R}$  is an obstacle function which is compatible with the homogeneous Dirichlet boundary condition in the sense that

$$\psi \leq 0 \text{ on } \Gamma_1. \quad (1.3)$$

Like problem (1.1), we will see that problem (1.2) is well-posed without a boundary condition along  $\Gamma_0$  when we seek solutions in suitable weighted Hölder or weighted Sobolev spaces.

In this article, we set  $n = 2$  and choose  $A$  to be the generator of the two-dimensional Heston stochastic volatility process with killing [29], a degenerate diffusion process well known in mathematical finance and a paradigm for a broad class of degenerate Markov processes, driven by  $n$ -dimensional Brownian motion, and corresponding generators which are degenerate elliptic integro-differential operators:

$$Av := -\frac{y}{2}(v_{xx} + 2\rho\sigma v_{xy} + \sigma^2 v_{yy}) - (r - q - y/2)v_x - \kappa(\theta - y)v_y + rv, \quad v \in C^\infty(\mathbb{H}). \quad (1.4)$$

Throughout this article, the coefficients of the *Heston operator*,  $A$ , are required to obey

**Assumption 1.1** (Ellipticity condition for the coefficients of the Heston operator). The coefficients defining  $A$  in (1.4) are constants obeying

$$\sigma \neq 0, -1 < \rho < 1, \quad (1.5)$$

and  $\kappa > 0$ ,  $\theta > 0$ ,  $r \geq 0$ , and  $q \geq 0$ .

For clarity of exposition in this article, we only consider the homogenous Dirichlet boundary condition  $u = 0$  on  $\Gamma_1$  in (1.1), as the modifications of our main results to include the case of a inhomogeneous Dirichlet boundary condition,  $u = g$  on  $\Gamma_1$  for some  $g : \mathcal{O} \cup \Gamma_1 \rightarrow \mathbb{R}$ , are straightforward and similar modifications are described in [15]; in problem (1.2), one requires that  $\psi : \mathcal{O} \cup \Gamma_1 \rightarrow \mathbb{R}$  be compatible with  $g$  in the sense that  $\psi \leq g$  on  $\Gamma_1$ .

**1.1. Summary of main results.** We shall state a selection of our main results here and then refer the reader to our guide to this article in §1.4 for more of our results on existence, uniqueness and regularity of solutions to variational equations and inequalities and corresponding obstacle problems. This article comprises part of the Ph.D thesis of C. Pop [47].

**1.1.1. Mathematical preliminaries.** As in [15, §2], we shall assume that the spatial domain has the following structure throughout this article:

**Definition 1.2** (Spatial domain for the Heston partial differential equation). Let  $\mathcal{O} \subset \mathbb{H}$  be a possibly unbounded domain with boundary  $\partial\mathcal{O}$ , let  $\Gamma_1 := \mathbb{H} \cap \partial\mathcal{O}$ , let  $\Gamma_0$  denote the interior of  $\{y = 0\} \cap \partial\mathcal{O}$ , and require that  $\Gamma_0$  is non-empty.

We write  $\partial\mathcal{O} = \Gamma_0 \cup \bar{\Gamma}_1 = \bar{\Gamma}_0 \cup \Gamma_1$  and note that the boundary portions  $\Gamma_0$  and  $\Gamma_1$  are relatively open in  $\partial\mathcal{O}$ . We shall consider weak solutions to (1.1) and (1.2), so we introduce our weighted Sobolev spaces. For  $1 \leq q < \infty$ , let

$$L^q(\mathcal{O}, \mathfrak{w}) := \{u \in L^1_{\text{loc}}(\mathcal{O}) : \|u\|_{L^q(\mathcal{O}, \mathfrak{w})} < \infty\}, \quad (1.6)$$

$$H^1(\mathcal{O}, \mathfrak{w}) := \{u \in L^2(\mathcal{O}, \mathfrak{w}) : (1+y)^{1/2}u, y^{1/2}|Du| \in L^2(\mathcal{O}, \mathfrak{w})\}, \quad (1.7)$$

$$H^2(\mathcal{O}, \mathfrak{w}) := \{u \in L^2(\mathcal{O}, \mathfrak{w}) : (1+y)^{1/2}u, (1+y)|Du|, y|D^2u| \in L^2(\mathcal{O}, \mathfrak{w})\}, \quad (1.8)$$

where  $Du = (u_x, u_y)$ ,  $D^2u = (u_{xx}, u_{xy}, u_{yx}, u_{yy})$ , all derivatives of  $u$  are defined in the sense of distributions, and

$$\|u\|_{L^q(\mathcal{O}, \mathfrak{w})}^q := \int_{\mathcal{O}} |u|^q \mathfrak{w} \, dx \, dy, \quad (1.9)$$

$$\|u\|_{H^1(\mathcal{O}, \mathfrak{w})}^2 := \int_{\mathcal{O}} (y|Du|^2 + (1+y)u^2) \mathfrak{w} \, dx \, dy, \quad (1.10)$$

$$\|u\|_{H^2(\mathcal{O}, \mathfrak{w})}^2 := \int_{\mathcal{O}} (y^2|D^2u|^2 + (1+y)^2|Du|^2 + (1+y)u^2) \mathfrak{w} \, dx \, dy, \quad (1.11)$$

with weight function  $\mathfrak{w} : \mathbb{H} \rightarrow (0, \infty)$  given by

$$\mathfrak{w}(x, y) := y^{\beta-1} e^{-\gamma|x|-\mu y}, \quad (x, y) \in \mathbb{H}, \quad (1.12)$$

where

$$\beta := \frac{2\kappa\theta}{\sigma^2} \quad \text{and} \quad \mu := \frac{2\kappa}{\sigma^2}, \quad (1.13)$$

and  $0 < \gamma < \gamma_0(A)$ , where  $\gamma_0$  depends only on the constant coefficients of  $A$  in (1.4). We call

$$\begin{aligned} a(u, v) := & \frac{1}{2} \int_{\mathcal{O}} (u_x v_x + \rho \sigma u_y v_x + \rho \sigma u_x v_y + \sigma^2 u_y v_y) y \mathfrak{w} \, dx \, dy \\ & - \frac{\gamma}{2} \int_{\mathcal{O}} (u_x + \rho \sigma u_y) v \operatorname{sign}(x) y \mathfrak{w} \, dx \, dy \\ & - \int_{\mathcal{O}} (a_1 y + b_1) u_x v \mathfrak{w} \, dx \, dy + \int_{\mathcal{O}} r u v \mathfrak{w} \, dx \, dy, \quad \forall u, v \in H^1(\mathcal{O}, \mathfrak{w}), \end{aligned} \quad (1.14)$$

the bilinear form associated with the Heston operator,  $A$ , in (1.4), noting that

$$a_1 := \frac{\kappa\rho}{\sigma} - \frac{1}{2} \quad \text{and} \quad b_1 := r - q - \frac{\kappa\theta\rho}{\sigma}. \quad (1.15)$$

We shall also avail of the

**Assumption 1.3** (Condition on the coefficients of the Heston operator). The coefficients defining  $A$  in (1.4) have the property that  $b_1 = 0$  in (1.15).

Assumption 1.3 involves no significant loss of generality because, using a simple affine changes of variables on  $\mathbb{R}^2$  which maps  $(\mathbb{H}, \partial\mathbb{H})$  onto  $(\mathbb{H}, \partial\mathbb{H})$  (see [15]), we can arrange that  $b_1 = 0$ .

The conditions (1.5) ensure that  $y^{-1}A$  is uniformly elliptic on  $\mathbb{H}$ . Indeed,

$$\frac{y}{2} (\xi_1^2 + 2\rho\sigma\xi_1\xi_2 + \sigma^2\xi_2^2) \geq \nu_0 y (\xi_1^2 + \xi_2^2), \quad \forall (\xi_1, \xi_2) \in \mathbb{R}^2, \quad (1.16)$$

where

$$\nu_0 := \min\{1, (1 - \rho^2)\sigma^2\}, \quad (1.17)$$

and  $\nu_0 > 0$  by Assumption 1.1.

Given a subset  $T \subseteq \partial\mathcal{O}$  we let  $H_0^1(\mathcal{O} \cup T, \mathfrak{w})$  be the closure in  $H^1(\mathcal{O}, \mathfrak{w})$  of  $C_0^\infty(\mathcal{O} \cup T)$ . Given a source function  $f \in L^2(\mathcal{O}, \mathfrak{w})$ , we call a function  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  a *solution to the variational equation* for the Heston operator with *homogeneous* Dirichlet boundary condition on  $\Gamma_1$  if

$$a(u, v) = (f, v)_{L^2(\mathcal{O}, \mathfrak{w})}, \quad \forall v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}). \quad (1.18)$$

If  $u \in H^2(\mathcal{O}, \mathfrak{w})$ , we recall from [15] that  $u$  is a solution to (1.1) if and only if  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  and  $u$  is a solution to (1.18).

We recall definition of the *Koch metric*,  $d$ , on  $\mathbb{H}$  introduced by Koch in [35, p. 11],

$$d(z, z_0) := \frac{|z - z_0|}{\sqrt{y + y_0 + |z - z_0|}}, \quad \forall z = (x, y), z_0 = (x_0, y_0) \in \bar{\mathbb{H}}, \quad (1.19)$$

where  $|z - z_0|^2 = (x - x_0)^2 + (y - y_0)^2$ . The metric  $d$  is equivalent to the *cycloidal metric* introduced by Daskalopoulos and Hamilton in [16, p. 901] for the study of the porous medium equation. For  $R > 0$  and  $z_0 \in \bar{\mathcal{O}}$ , we denote

$$B_R(z_0) = \{z \in \mathcal{O} : d(z, z_0) < R\}, \quad (1.20)$$

while

$$\bar{B}_R(z_0) = \{z \in \bar{\mathcal{O}} : d(z, z_0) \leq R\},$$

is the usual closure of  $B_R(z_0)$  in  $\bar{\mathcal{O}}$ .

We say that a domain,  $U \subset \mathbb{H}$ , obeys an *exterior cone condition relative to  $\mathbb{H}$  at a point  $z_0 \in \partial U$*  if there exists a finite, right circular cone  $K = K_{z_0} \subset \bar{\mathbb{H}}$  with vertex  $z_0$  such that  $\bar{U} \cap K_{z_0} = \{z_0\}$  (compare [28, p. 203]). A domain,  $U$ , obeys a *uniform exterior cone condition relative to  $\mathbb{H}$  on  $T \subset \partial U$*  if  $U$  satisfies an exterior cone condition relative to  $\mathbb{H}$  at every point  $z_0 \in T$  and the cones  $K_{z_0}$  are all congruent to some fixed finite cone,  $K$  (compare [28, p. 205]). Recall that  $\Gamma_0$  is the interior of the portion,  $\bar{\mathcal{O}} \cap \partial\mathbb{H}$ , of the boundary,  $\partial\mathcal{O}$ , of the domain  $\mathcal{O} \subseteq \mathbb{H}$ .

**Definition 1.4** (Interior and exterior cone conditions). Let  $K$  be a finite, right circular cone. We say that  $\mathcal{O}$  obeys *interior and exterior cone conditions at  $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$  with cone  $K$*  if the domains  $\mathcal{O}$  and  $\mathbb{H} \setminus \bar{\mathcal{O}}$  obey exterior cone conditions relative to  $\mathbb{H}$  at  $z_0$  with cones congruent to  $K$ . We say that  $\mathcal{O}$  obeys *uniform interior and exterior cone conditions on  $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$  with cone  $K$*  if the domains  $\mathcal{O}$  and  $\mathbb{H} \setminus \bar{\mathcal{O}}$  obey exterior cone conditions relative to  $\mathbb{H}$  at each point  $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$  with cones congruent to  $K$ .

**1.1.2. Boundary local supremum bounds.** In the statement of the supremum estimates, we use the following

**Definition 1.5** (Volume of sets). If  $S \subset \bar{\mathbb{H}}$  is a Borel measurable subset, we let  $|S|_\beta$  denote the volume of  $S$  with respect to the measure  $y^\beta dx dy$ , and  $|S|_\mathfrak{w}$  denote the volume of  $S$  with respect to the measure  $\mathfrak{w} dx dy$ .

**Remark 1.6.** As in [16, Theorem I.1.1], the assumption that  $\kappa > 0$  and  $\theta > 0$ , that is, that the coefficient of  $v_y$  in the definition (1.4) of  $-A$ , is strictly positive is of crucial importance. We notice from (1.13) that  $\beta > 0$  and so, the weight  $\mathfrak{w}$  belongs to  $L^1(\mathbb{H})$ . Therefore, the volume of balls  $\mathbb{B}_R(z_0)$  centered at points  $z_0 \in \Gamma_0$  is finite with respect to the weights  $y^{\beta-1} dx dy$ , and  $\mathfrak{w} dx dy$ , a fact which we repeatedly use in this article. Clearly, if  $\beta$  were negative, then  $\mathfrak{w}$  belongs to  $L_{\text{loc}}^1(\mathbb{H})$ , but not to  $L^1(\mathbb{H})$ .

We have the following analogue of [35, Proposition 4.5.1] and [28, Theorem 8.15].

**Theorem 1.7** (Supremum estimates at points in  $\bar{\Gamma}_0$ ). *Let  $K$  be a finite, right circular cone and let  $\mathcal{O}$  be a domain obeying the uniform interior cone condition on  $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$  with cone  $K$ . Let  $s > n + \beta$ . Then there are positive constants  $C$  and  $\bar{R}$ , depending at most on the coefficients of the Heston operator,  $A$ , and on  $K$ ,  $n$  and  $s$ , such that for any  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  obeying (1.18) with source function  $f \in L^s(B_{\bar{R}}(z_0), y^{\beta-1}) \cap L^2(\mathcal{O}, \mathfrak{w})$ , the following holds: for all  $z_0 \in \bar{\Gamma}_0$  and all  $R$  such that  $0 < 2R \leq \bar{R}$ , we have*

$$\operatorname{ess\,sup}_{B_R(z_0)} |u| \leq C \left( |B_{2R}(z_0)|_{\beta-1}^{-1/2} \|u\|_{L^2(B_{2R}(z_0), y^{\beta-1})} + \|f\|_{L^s(B_{2R}(z_0), y^{\beta-1})} \right). \quad (1.21)$$

**Remark 1.8** (Use of the weight  $y^{\beta-1}$  versus  $\mathfrak{w}$  in Theorem 1.7). Notice that on the right-hand-side of estimate (1.21) we have  $\|f\|_{L^s(B_{2R}(z_0), y^{\beta-1})}$  instead of  $\|f\|_{L^s(B_{2R}(z_0), \mathfrak{w})}$ . This allows us to conclude that the constant  $C$  appearing in (1.21) is independent of the point  $z_0 \in \bar{\Gamma}_0$ . By (1.12), the weight  $\mathfrak{w}$  contains the term  $e^{-\gamma|x|}$ , which means that the constant  $C$  will depend on the  $x$ -coordinate of the point  $z_0 \in \bar{\Gamma}_0$ , if we replace  $\|f\|_{L^s(B_{2R}(z_0), y^{\beta-1})}$  by  $\|f\|_{L^s(B_{2R}(z_0), \mathfrak{w})}$  on the right-hand-side of (1.21).

**Remark 1.9** (Hypothesis on the domain). In the statement of Theorem 1.7, instead of assuming that the domain  $\mathcal{O}$  satisfies the uniform interior cone condition on  $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$  (Definition 1.4), we could have assumed the weaker condition (4.1). It is sufficient that condition (4.1) is obeyed locally, that is the constants  $\bar{R}$  and  $c$  may depend on  $z_0 \in \bar{\Gamma}_0$ . In this case, the constants  $C$  and  $\bar{R}$  appearing in the statement of Theorem 1.7 will depend on  $z_0 \in \bar{\Gamma}_0$  as well.

**Remark 1.10** (Role of the Dirichlet boundary condition in Theorem 1.7). The homogeneous Dirichlet boundary condition along  $\Gamma_1$  satisfied by  $u$ , that is  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ , does not play any role in establishing the supremum estimate (1.21) at points  $z_0 \in \Gamma_0$ , such that  $B_{2R}(z_0) \subset \bar{\mathcal{O}}$ .

1.1.3. *Hölder continuity up to the boundary for solutions to the variational equation.* For  $z_0 \in \bar{\mathcal{O}}$  and  $R > 0$ , we denote

$$M_R := \operatorname{ess\,sup}_{B_R(z_0)} u(z), \quad (1.22)$$

$$m_R := \operatorname{ess\,inf}_{B_R(z_0)} u(z), \quad (1.23)$$

and we let

$$\operatorname{osc}_{B_R(z_0)} u := M_R - m_R$$

denote the oscillation of  $u$  over the ball  $B_R(z_0)$ . From Theorem 1.7, we know that  $M_R$  and  $m_R$  are finite quantities and  $\operatorname{osc}_{B_R(z_0)} u$  is well-defined for weak solutions  $u$  as in Theorem 1.7.

Following [1, §1.26], for a domain  $U \subset \mathbb{H}$ , we let  $C(U)$  denote the vector space of continuous functions on  $U$  and let  $C(\bar{U})$  denote the Banach space of functions in  $C(U)$  which are bounded and uniformly continuous on  $U$ , and thus have unique bounded, continuous extensions to  $\bar{U}$ , with norm

$$\|u\|_{C(\bar{U})} := \sup_{\bar{U}} |u|.$$

Given  $\alpha \in (0, 1)$ , we say that  $u \in C_s^\alpha(\bar{U})$  if  $u \in C(\bar{U})$  and

$$\|u\|_{C_s^\alpha(\bar{U})} < \infty,$$

where

$$\|u\|_{C_s^\alpha(\bar{U})} := [u]_{C_s^\alpha(\bar{U})} + \|u\|_{C(\bar{U})}, \quad (1.24)$$

and

$$[u]_{C_s^\alpha(\bar{U})} := \sup_{\substack{z_1, z_2 \in U \\ z_1 \neq z_2}} \frac{|u(z_1) - u(z_2)|}{d(z_1, z_2)^\alpha}. \quad (1.25)$$

Moreover,  $C_s^\alpha(\bar{U})$  is a Banach space [16, §I.1] with respect to the norm (1.24). We say that  $u \in C_s^\alpha(U)$  if  $u \in C_s^\alpha(\bar{V})$  for all precompact open subsets  $V \Subset U \cup \Gamma_0$ .

When  $U$  may be *unbounded*, we let  $C_{\text{loc}}(\bar{U})$  denote the linear subspace of functions  $u \in C(U)$  such that  $u \in C(\bar{V})$  for every precompact open subset  $V \Subset \bar{U}$ ; similarly, we let  $C_{s, \text{loc}}^\alpha(\bar{U})$  denote the linear subspace of functions  $u \in C_s^\alpha(U)$  such that  $u \in C_s^\alpha(\bar{V})$  for every precompact open subset  $V \Subset \bar{U}$ .

For any non-negative integer  $k$ , we let  $C_0^k(U \cup \Gamma_0)$  denote the linear subspace of functions  $u \in C^k(U)$  such that  $u \in C^k(\bar{V})$  for every precompact open subset  $V \Subset U \cup \Gamma_0$  and similarly define  $C_0^\infty(U \cup \Gamma_0)$ .

We have the following analogue of [28, Theorem 8.27 & 8.29] and [35, Theorem 4.5.5 & 4.5.6] for the boundary portion  $\Gamma_0$ .

**Theorem 1.11** (Hölder continuity up to  $\bar{\Gamma}_0$  for solutions to the variational equation). *Let  $K$  be a finite, right circular cone, let  $\mathcal{O}$  be a domain, let  $s > \max\{2n, n + \beta\}$ , and let  $\bar{R}_0$  be a positive constant. Let  $f \in L^2(\mathcal{O}, \mathfrak{w})$  and  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  obey (1.18). Suppose  $z_0 \in \bar{\Gamma}_0$ , where  $\mathcal{O}$  obeys an interior and exterior cone condition with cone  $K$  at  $z_0$  and a uniform exterior cone condition with cone  $K$  along  $\bar{\Gamma}_1 \cap \bar{B}_{\bar{R}_0}(z_0)$  if  $\Gamma_1 \cap \bar{B}_{\bar{R}_0}(z_0) \neq \emptyset$ , and that*

$$f \in L^s(B_{\bar{R}_0}(z_0), y^{\beta-1}). \quad (1.26)$$

*Then there are a positive constant  $\bar{R}$  depending at most on  $\bar{R}_0, n, \beta$  and  $K$ , and positive constants  $C, C_1$  depending at most on the coefficients of the Heston operator,  $A$ , together with  $\bar{R}_0, n, s$  and  $K$ , and*

$$\|f\|_{L^s(B_{\bar{R}_0}(z_0), y^{\beta-1})} \quad \text{and} \quad \|u\|_{L^\infty(B_{\bar{R}_0}(z_0))},$$

*and a constant  $\alpha_0 \in (0, 1)$ , depending at most on  $s, n$  and  $\beta$ , and a constant  $\alpha_1 \in (0, 1)$  depending in addition on the coefficients of the Heston operator,  $A$ , together with  $\bar{R}_0$  and  $K$ , if  $\Gamma_1 \cap \bar{B}_{\bar{R}_0}(z_0) \neq \emptyset$ , such that the following holds. For all  $R$  such that  $0 < 8R \leq \bar{R}$ , we have*

$$\text{osc}_{B_R(z_0)} u \leq CR^{\alpha_0}, \quad (1.27)$$

*and we have  $u \in C_s^{\alpha_1}(\bar{B}_{\bar{R}}(z_0))$  with*

$$|u(z_1) - u(z_2)| \leq C_1 d(z_1, z_2)^{\alpha_1}, \quad \forall z_1, z_2 \in \bar{B}_{\bar{R}}(z_0). \quad (1.28)$$

For any  $\delta > 0$ , we let

$$\mathcal{O}_\delta := \mathcal{O} \cap (\mathbb{R} \times (0, \delta)). \quad (1.29)$$

We then have the

**Corollary 1.12** (Hölder continuity up to  $\bar{\Gamma}_0$  for solutions to the variational equation). *Let  $K$  be a finite, right circular cone, let  $\mathcal{O}$  be a domain, let  $s > \max\{2n, n + \beta\}$ , and let  $\delta$  be a positive constant. Assume that  $\mathcal{O}$  obeys a uniform interior and exterior cone condition with cone  $K$  on  $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$  and a uniform exterior cone condition with cone  $K$  along  $\Gamma_1 \cap \partial \mathcal{O}_\delta$ . Let  $f \in L^2(\mathcal{O}, \mathfrak{w})$  and  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  obey (1.18) and assume that  $f$  and  $u$  obey*

$$\sup_{z_0 \in \Gamma_0} \|f\|_{L^s(B_\delta(z_0), y^{\beta-1})} < \infty \quad \text{and} \quad \sup_{z_0 \in \Gamma_0} \|u\|_{L^2(B_\delta(z_0), y^{\beta-1})} < \infty. \quad (1.30)$$



Then there are a constant  $\delta_1$ , depending only on  $\delta$ , and a constant  $\alpha_1 \in (0, 1)$ , depending at most on the coefficients of the Heston operator,  $A$ , together with  $\delta$ ,  $n$ ,  $s$ , and  $K$  such that

$$u \in C_s^{\alpha_1}(\bar{\mathcal{O}}_{\delta_1}).$$

Moreover,  $\|u\|_{C_s^{\alpha_1}(\bar{\mathcal{O}}_{\delta_1})}$  is bounded by a constant depending at most on the coefficients of the Heston operator,  $A$ , together with  $n$ ,  $s$ ,  $K$ ,  $\delta$  and the supremum bounds in (1.30).

The constant  $\delta_1$  in Corollary 1.12 depends on  $\delta$  through the geometry of the balls  $B_\delta(z_0)$ ,  $z_0 \in \Gamma_0$ , defined by the Koch metric,  $d$ . We choose  $\delta_1$  so that we can cover the strip  $\mathcal{O}_{\delta_1}$  by balls  $B_\delta(z_0)$ ,  $z_0 \in \Gamma_0$ . By Lemma 2.4, we see that we can find a small enough constant  $c$ , say  $c = 1/4000$ , and a sequence of points  $\{z_0^n\}_{n \in \mathbb{N}} \subset \Gamma_0$ , such that

$$\mathcal{O}_{c\delta^2} \subset \bigcup_{n \in \mathbb{N}} E_{2c\delta^2}(z_0^n) \subset \bigcup_{n \in \mathbb{N}} B_\delta(z_0^n),$$

where  $E_r(z)$  denotes the Euclidean ball relative to the domain  $\mathcal{O}$ , of radius  $r$  and with center  $z$  (see Definition 2.3). For the second inclusion above, we have used (2.7) and the fact that we choose the constant  $c$  small enough, independent of  $\delta$ . Therefore, the constant  $\delta_1$  in the statement of Corollary 1.12 is chosen so that  $\delta_1 \leq c\delta^2$ . For simplicity, we do not describe the dependency of  $\delta_1$  on  $\delta$  in the statement of Corollary 1.12.

Condition (1.30) on  $u$  is satisfied when  $u \in L^2(\mathcal{O}, \mathfrak{w})$  and the domain  $\mathcal{O}$  is bounded in the  $x$ -direction, as we can see from the definition (1.12) of the weight  $\mathfrak{w}$ . We shall assume the condition (1.30) on  $u$  to ensure that  $u$  is a bounded function on  $\mathcal{O}_{\delta_1}$  via Theorem 1.7.

**Remark 1.13** (Hypothesis on the domain). To establish (1.27) and (1.28), instead of assuming that the domain  $\mathcal{O}$  satisfies the uniform interior and exterior cone condition on  $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ , we could have assumed the weaker conditions (4.1), at points  $z_0 \in \bar{\Gamma}_0$ , and (5.60), at points  $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$ .

**Remark 1.14** (Comparison with the case of a portion of the boundary where the operator is non-degenerate). The term  $\sigma(\sqrt{RR_0})$ , where  $\sigma(R) := \text{osc}_{\partial\mathcal{O} \cap \bar{B}_R(z_0)} u$ , which appears in [28, Equation (8.72)] in the statement of [28, Theorem 8.27] does not appear in the statement of our Theorem 1.11. The reason is that unlike in [28, Equation (8.71)], the test functions defined in the proof of Theorem 1.11 do not need to involve  $\text{ess sup}_{\partial\mathcal{O} \cap \bar{B}_R(z_0)} u$  or  $\text{ess inf}_{\partial\mathcal{O} \cap \bar{B}_R(z_0)} u$  since no boundary condition is imposed on  $v$  along  $\Gamma_0$ , in contrast to the Dirichlet boundary condition assumed for  $v$  in the proofs of [28, Theorem 8.18 & 8.26].

**1.1.4. Hölder continuity up to the boundary for solutions to the variational inequality.** Given a source function  $f \in L^2(\mathcal{O}, \mathfrak{w})$  and an obstacle function  $\psi \in H^1(\mathcal{O}, \mathfrak{w})$  obeying (1.3) in the sense that

$$\psi^+ \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}), \tag{1.31}$$

we call  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  a solution to the variational inequality for the Heston operator with homogeneous Dirichlet boundary condition along  $\Gamma_1$  if

$$\begin{aligned} a(u, v - u) &\geq (f, v - u)_{L^2(\mathcal{O}, \mathfrak{w})} \quad \text{and} \quad u \geq \psi \text{ a.e. on } \mathcal{O}, \\ \forall v &\in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}) \text{ with } v \geq \psi \text{ a.e. on } \mathcal{O}. \end{aligned} \tag{1.32}$$

Given additional mild conditions on  $f$  and  $\psi$ , we prove in [15] that there is a unique solution,  $H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ , to (1.32). For Theorem 1.16, we require



**Hypothesis 1.15** (Conditions on the source and obstacle functions). For some  $\delta > 0$ ,

$$f \in L^2(\mathcal{O}, \mathfrak{w}) \cap L^\infty(\mathcal{O}_\delta), \quad (1.33)$$

$$\psi \in H^2(\mathcal{O}_\delta, \mathfrak{w}) \cap L^\infty(\mathcal{O}_\delta), \quad (1.34)$$

where  $\mathcal{O}_\delta$  is defined in (1.29).

We then have

**Theorem 1.16** (Hölder continuity up to  $\bar{\Gamma}_0$  for solutions to the variational inequality). *Require that  $\mathcal{O}$  obeys a uniform interior and exterior cone condition on  $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$  with cone  $K$  and a uniform exterior cone condition with cone  $K$  along  $\Gamma_1 \cap \partial\mathcal{O}_\delta$ . Assume that  $f$  obeys (1.33), that  $\psi$  obeys (1.31) and (1.34), and that*

$$\operatorname{ess\,sup}_{\mathcal{O}_\delta} (A\psi - f)^+ < \infty. \quad (1.35)$$

Let  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  be a solution to (1.32) such that at least one of the following conditions holds,

$$\operatorname{height}(\mathcal{O}) < \infty \quad \text{or} \quad u \in W^{1,\infty}(\mathcal{O}_\delta \setminus \mathcal{O}_{\delta/2}), \quad (1.36)$$

where  $\delta$  is as in Hypothesis 1.15. Then

$$u \in C_s^{\alpha_1}(\bar{\mathcal{O}}_{\delta_1}),$$

where  $\alpha_1 \in (0, 1)$  and  $\delta_1$  are as in Corollary 1.12.

**Remark 1.17** (Hypotheses on the solution to the variational inequality). The second condition in (1.36) in Theorem 1.16 is implied by the  $W_{\operatorname{loc}}^{2,p}(\mathcal{O})$  regularity result [15, Theorem 6.18] for  $p > 2$  and corresponding  $W^{2,p}(U)$  a priori estimates using the conditions (1.33) and (1.34), and the Sobolev embedding  $W^{2,p}(U) \hookrightarrow C_b^1(U)$  for domains  $U \Subset \mathbb{H}$  with the interior cone property [1, Theorem 5.4 (C)].

1.1.5. *Harnack inequality for non-negative solutions to the variational equation.* We also have the following analogue of [28, Theorem 8.20] and [35, Theorem 4.5.3].

**Theorem 1.18** (Harnack inequality near  $\Gamma_0$ ). *Let  $\bar{R}$  be a positive constant. Then there is a positive constant  $C$ , depending at most on the coefficients of the Heston operator,  $A$ , together with  $n$  and  $\bar{R}$ , such that for any non-negative  $u \in H^1(\mathcal{O}, \mathfrak{w})$  obeying (1.18) with  $f = 0$  on  $B_{\bar{R}}(z_0)$ , we have*

$$\sup_{B_R(z_0)} u \leq C \inf_{B_R(z_0)} u, \quad (1.37)$$

for all  $z_0 \in \Gamma_0$  and  $0 < 4R \leq \min\{\bar{R}, \operatorname{dist}(z_0, \Gamma_1)\}$ .

**1.2. Survey of previous research.** We provide a brief survey of some work by other authors on supremum bounds, Harnack inequalities, and Hölder continuity of weak solutions to degenerate elliptic partial differential equations most closely related to the results described in our article. We cannot appeal directly to results described in the cited references in order to derive supremum bounds, a Harnack inequality, and Hölder continuity near the boundary for weak solutions to the Heston equation because of differences between the following principal features in the cited references and those in our article:

- (1) Structure of the differential operators, including the nature of the degeneracy and presence of lower-order terms;

- (2) Boundary conditions, where in our article no boundary condition is specified on  $\Gamma_0$  (the “degenerate” portion of the boundary  $\partial\mathcal{O}$ ) and a Dirichlet condition is prescribed along  $\Gamma_1$  (the “non-degenerate” portion of the boundary  $\partial\mathcal{O}$ );
- (3) Weights used to define weighted Sobolev spaces and weak solutions;
- (4) Dependency of the constants in estimates, with those appearing in our estimates depending at most on the  $L^q(B \cap \mathcal{O}, \mathfrak{w})$  norm ( $q > 2$ ) of  $f$  on neighborhoods  $B$  of boundary points, the  $L^2(\mathcal{O}, \mathfrak{w})$  norm of  $u$ , the geometry of  $\Gamma_1$ , and the constant coefficients of the Heston operator,  $A$ .

Furthermore, near  $\Gamma_0$ , the weights  $y\mathfrak{w}$  and  $(1+y)\mathfrak{w}$  used in our definition of  $H^1(\mathcal{O}, \mathfrak{w})$  are  $y^\beta e^{-\gamma|x|-\mu y}$  and  $y^{\beta-1} e^{-\gamma|x|-\mu y}$ , respectively, with constants  $0 < \beta, \mu < \infty$  depending on the coefficients of  $u_{yy}$  and  $u_y$  when our differential operator  $A$  is expressed in standard divergence form and so the weight  $\mathfrak{w}$  depends on both the second and first-order parts of  $A$  and not just on the second-order part of the differential operator, unlike in the cited references. Note also that  $\mathfrak{w}$  is zero along  $\Gamma_0$ , where  $A$  is degenerate, but positive along  $\Gamma_1$ , where  $A$  is non-degenerate.

Koch [35] considers certain linear elliptic and parabolic degenerate model partial differential equations in divergence form, with a degeneracy similar to ours, and which arise as linearizations of the porous medium equation. However, while Koch uses Sobolev weights which are comparable to ours, his methods (which use pointwise estimates for fundamental solutions and Moser iteration) are different to ours (which use Moser iteration and the abstract John-Nirenberg inequality). Moreover, he does not consider the case where  $\partial\mathcal{O} = \Gamma_0 \cup \Gamma_1$ , where  $A$  is degenerate along  $\Gamma_0$  but non-degenerate along  $\Gamma_1$ . Finally, Koch does not consider applications to Hölder regularity of solutions to variational inequalities as we do in our article.

*1.2.1. Supremum bounds for weak solutions to degenerate partial differential equations.* One of the earliest and best known articles on supremum bounds near the domain boundary for weak solutions to certain linear degenerate elliptic partial differential equations is due to Murthy and Stampacchia [44, 45]; more recently, supremum bounds for weak solutions to certain quasi-linear degenerate elliptic and parabolic partial differential equations have been established by Amanov and Mamedov [2], Bonafede and Nicolosi [6, 7], Borsuk [8], and Cianci [12, 13]. Koch [35] obtained supremum bounds for his linear degenerate parabolic model partial differential equation in [35, Proposition 4.5.1].

*1.2.2. Harnack inequality for weak solutions to degenerate partial differential equations.* Well-known early results on the Harnack inequality near the domain boundary for weak solutions to certain linear degenerate elliptic partial differential equations are due to Chanillo and Wheeden [10], Fabes, Kenig, and Serapioni [20, 21], Franchi and Serapioni [27], and, more recently, Cruz-Uribe, Di Gironimo, and Sbordone [14]. Koch obtains a Harnack inequality for his linear degenerate parabolic model partial differential equation in [35, Theorem 4.5.3]. Harnack inequalities for weak solutions to certain quasi-linear degenerate elliptic partial differential equations have been obtained by Amanov and Mamedov [39], Di Fazio, Fanciullo, and Zamboni [23, 22, 55], Mohammed [43], Pinggen [46], Sawyer and Wheeden [48], Stredulinsky [49], and Wang, Wang, Yin, and Zhou [54]. We note that Kinnunen and Kuusi also appealed to the abstract John-Nirenberg inequality and a weighted Poincaré inequality to prove a Harnack inequality for solutions to non-linear, second-order parabolic equations [34].

*1.2.3. Hölder continuity for weak solutions to degenerate partial differential equations.* Early results on Hölder continuity up to the boundary for weak solutions to certain linear degenerate elliptic partial differential equations are due to Fabes, Kenig, and Serapioni [20, 21], Franchi and

Serapioni [27], Murthy and Stampacchia [44, 45], and more recently, Cruz-Uribe, Di Gironimo, and Sbordone [14], Duc, Phuc, and Nguyen [18]. Results on Hölder continuity up to the boundary for weak solutions to certain quasi-linear degenerate elliptic partial differential equations are due to Amanov and Mamedov [39], Di Fazio, Fanciullo, and Zamboni [23, 22, 55], Pingren [46], Stredulinsky [49], and Wang, Wang, Yin, and Zhou [54]. Koch proves Hölder continuity for weak solutions for his linear degenerate parabolic model partial differential equation in [35, Theorem 4.5.5].

**1.2.4. Hölder continuity for weak solutions to degenerate obstacle problems.** For variational inequalities defined by degenerate elliptic or parabolic operators, there has been little previous research that we are aware of which concerns boundary regularity of solutions, though Mastroeni and Matzeu [41, 42] and, more recently, Vitanza and Zamboni [52, 53] describe existence and uniqueness results for solutions in certain weighted Sobolev spaces.

**1.3. Extensions to degenerate operators in higher dimensions.** The Heston stochastic volatility process and its associated generator serve as paradigms for degenerate Markov processes and their degenerate elliptic generators which appear widely in mathematical finance.

**1.3.1. Degenerate diffusion processes and partial differential operators.** Generalizations of the Heston process to higher-dimensional, degenerate diffusion processes may be accommodated by extending the framework developed in this article and we shall describe extensions in a sequel. First, the two-dimensional Heston process has natural  $d$ -dimensional analogues [26] defined, for example, by coupling non-degenerate  $(d - 1)$ -diffusion processes with degenerate one-dimensional processes [11, 40, 56]. Elliptic differential operators arising in this way have time-independent, affine coefficients but, as one can see from standard theory [28, 36, 37, 38] and previous work of Daskalopoulos and her collaborators [16, 17] on the porous medium equation, we would not expect significant new difficulties to arise when extending the methods and results of this article to the case of elliptic and parabolic operators in higher dimensions and variable coefficients, depending on both spatial variables or time and possessing suitable regularity and growth properties.

Specifically, we expect that all of the main results of this article should extend to the case of a degenerate elliptic operator on a subdomain  $\mathcal{O}$  of a half-space  $\mathbb{H} := \mathbb{R}^{n-1} \times (0, \infty)$ ,

$$Av := -x_n a_{ij} v_{x_i x_j} - b_i v_{x_i} + cv, \quad v \in C^\infty(\mathcal{O}),$$

under the assumptions that the matrix  $(a_{ij})$  is strictly elliptic,  $b_n \geq \nu > 0$ , for some constant  $\nu > 0$ , and  $c \geq 0$  and the coefficients have suitable growth and regularity properties. See [25] for an analysis with applications to probability theory based on a parabolic version of this type of elliptic operator as well as [24] for weak maximum principles for a general class of degenerate elliptic operators.

**1.3.2. Degenerate Markov processes and partial-integro differential operators.** The Heston process also has natural extensions to  $d$ -dimensional degenerate affine jump-diffusion processes with Markov generators which are degenerate elliptic partial-integro differential operators. A well-known example of such a two-dimensional process is due to Bates [3] and the definition of this process has been extended to higher dimensions by Duffie, Pan, and Singleton [19]. Stationary jump diffusion processes of this kind and their partial-integro differential operator generators naturally lie within the framework of Feller processes and Feller generators [30, 31, 32], where the non-local nature of the partial-integro differential operators provides new challenges when considering obstacle problems; see [9] for recent research by Caffarelli and Figalli in this direction, as well as that of Bayraktar and Xing [4].

**1.4. Mathematical highlights and guide to the article.** For the convenience of the reader, we provide a brief outline of the article. We begin in §2 by describing a Sobolev inequality due to H. Koch [35] and prove a Poincaré inequality for our weighted Sobolev spaces. In §3, we recall the abstract John-Nirenberg inequality (Theorem 3.1) due to E. Bombieri and E. Giusti [5] and justify its application (via Proposition 3.2) in the setting of our weighted Sobolev spaces. The supremum estimate near  $\bar{\Gamma}_0$  for solutions to the variational equation (1.18) (Theorem 1.7) is proved in §4 by adapting the Moser iteration technique employed in the proof of [28, Theorem 8.15] to the setting of our degenerate elliptic operators and weighted Sobolev spaces. Section 5 contains our proof of local Hölder continuity along  $\bar{\Gamma}_0$  of solutions to the variational equation (1.18) (Theorem 1.11). The essential difference between the proof of Theorem 1.11 and the proof of its classical analogue for weak solutions to non-degenerate elliptic equations [28, Theorems 8.27 & 8.29] consists in a modification of the methods of [28, §8.6, §8.9, & §8.10] when deriving our energy estimates (5.12), where we adapt the application of the John-Nirenberg inequality and Poincaré inequality to our framework of weighted Sobolev spaces. In §5, we apply the penalization method and techniques of [15], together with Theorem 1.11, to prove local Hölder continuity along  $\bar{\Gamma}_0$  of solutions to the variational inequality (1.2) (Theorem 1.16). Finally, in §7 we prove the Harnack inequality (Theorem 1.18) for solutions to the variational equation (1.18). Appendix A contains the proofs of auxiliary results employed throughout the article whose proofs are sufficiently technical that they would have otherwise interrupted the logical flow of our article.

**1.5. Notation and conventions.** In the definition and naming of function spaces, including spaces of continuous functions, Hölder spaces, or Sobolev spaces, we follow Adams [1] and alert the reader to occasional differences in definitions between [1] and standard references such as Gilbarg and Trudinger [28] or Krylov [36, 37]. We denote  $\mathbb{R}_+ := (0, \infty)$ ,  $\bar{\mathbb{R}}_+ := [0, \infty)$ ,  $\mathbb{H} := \mathbb{R} \times \mathbb{R}_+$ , and  $\bar{\mathbb{H}} := \mathbb{R} \times \bar{\mathbb{R}}_+$ . We let  $\mathbb{N} := \{1, 2, 3, \dots\}$  denote the set of positive integers. For  $x, y \in \mathbb{R}$ , we denote  $x \wedge y := \min\{x, y\}$ ,  $x \vee y := \max\{x, y\}$ . Moreover,  $x^+ := x \vee 0$  and  $x^- := -(x \wedge 0)$ , so  $x = x^+ - x^-$  and  $|x| = x^+ + x^-$ , a convention which differs from that of [28, §7.4].

Throughout our article, we fix  $n = 2$ . We keep track of the dependency of many of our estimates on the dimension,  $n$ , of  $\mathbb{H} = \mathbb{R}^{n-1} \times (0, \infty)$  in our analysis, even though  $n = 2$  in this article, as this will make it easier to extend our results to partial differential equations on domains in  $\mathbb{H}$  which preserve the key features of (1.1).

When we label a condition an *Assumption*, then it is considered to be universal and in effect throughout this article and so not referenced explicitly in theorem and similar statements; when we label a condition a *Hypothesis*, then it is only considered to be in effect when explicitly referenced.

## 2. SOBOLEV AND POINCARÉ INEQUALITIES FOR WEIGHTED SOBOLEV SPACES

We review a Sobolev inequality (Lemma 2.2) due to H. Koch [35] and prove a Poincaré inequality (Lemma 2.5) for weighted Sobolev spaces. Recall from [35, Corollary 4.3.4] that the weight  $y^{\beta-1}$  defines a doubling measure,  $y^{\beta-1} dx dy$  on  $\mathbb{H}$  for any  $\beta > 0$  (see, for example, [51, Definition 1.2.6]), where  $dx dy$  is Lebesgue measure on  $\mathbb{H}$ . In the following Lemma 2.2 and the sequel, we will need the following

**Definition 2.1.** Throughout our article, we fix

$$p = \frac{2(n + \beta)}{n + \beta - 1}, \tag{2.1}$$

for any  $\beta > 0$ .

**Lemma 2.2** (Weighted Sobolev inequality). [35, Lemma 4.2.4] *Let  $p$  be as in (2.1). Then there is a positive constant  $C = C(n, p)$  such that*

$$\int_{\mathbb{H}} |u|^p y^{\beta-1} dx dy \leq c \left( \int_{\mathbb{H}} |u|^2 y^{\beta-1} dx dy \right)^{\frac{p-2}{2}} \int_{\mathbb{H}} |\nabla u|^2 y^{\beta} dx dy, \quad (2.2)$$

for any  $u \in L^2(\mathbb{H}, y^{\beta-1})$  such that  $\nabla u \in L^2(\mathbb{H}, y^{\beta})$ .

For  $R > 0$  and  $z_0 \in \bar{\mathbb{H}}$ , we denote

$$\mathbb{B}_R(z_0) = \{z \in \mathbb{H} : d(z, z_0) < R\}, \quad (2.3)$$

while

$$\bar{\mathbb{B}}_R(z_0) = \{z \in \bar{\mathbb{H}} : d(z, z_0) \leq R\},$$

is the usual closure of  $\mathbb{B}_R(z_0)$  in  $\bar{\mathbb{H}}$ . Notice that in definition (1.20),  $B_R(z_0)$  denotes the ball relative to the domain  $\mathcal{O}$ , while  $\mathbb{B}_R(z_0)$  denotes the ball relative to the half-space,  $\mathbb{H}$ .

**Definition 2.3** (Balls with respect to the Euclidean metric). Given  $z_0 \in \bar{\mathbb{H}}$  and  $R > 0$ , let

$$\mathbb{E}_R(z_0) := \{z \in \mathbb{H} : |z - z_0| < R\}, \quad (2.4)$$

$$E_R(z_0) := \{z \in \mathcal{O} : |z - z_0| < R\}. \quad (2.5)$$

**Lemma 2.4.** [35, Lemma 4.3.3] *There is a positive constant  $c$ , depending only on  $n$  and  $\beta$ , such that, for any  $R > 0$  and  $z_0 \in \bar{\mathbb{H}}$ ,*

$$c^{-1} R^n (R + \sqrt{y_0})^{n+2\beta} \leq |\mathbb{B}_R(z_0)|_{\beta} \leq c R^n (R + \sqrt{y_0})^{n+2\beta}. \quad (2.6)$$

Moreover, the following inclusions hold,

$$\mathbb{E}_{R_1}(z_0) \subseteq \mathbb{B}_R(z_0) \subseteq \mathbb{E}_{R_2}(z_0), \quad (2.7)$$

where  $R_1 = R(R + \sqrt{y_0})/2000$  and  $R_2 = R(R + 2\sqrt{y_0})$ .

**Lemma 2.5** (Poincaré inequality). *Let  $z_0 \in \partial\mathbb{H}$  and  $R > 0$ . Then there is a positive constant  $C$ , depending on  $\beta$ ,  $n$  and  $R$ , such that for any  $u \in H^1(\mathbb{B}_R(z_0), \mathfrak{w})$ , we have*

$$\inf_{c \in \mathbb{R}} \left( \int_{\mathbb{B}_R(z_0)} |u(z) - c|^2 y^{\beta-1} dx dy \right)^{1/2} \leq C \left( \int_{\mathbb{B}_R(z_0)} |\nabla u(z)|^2 y^{\beta} dx dy \right)^{1/2}. \quad (2.8)$$

**Corollary 2.6** (Poincaré inequality with scaling). *There is a positive constant  $C$ , depending only on  $\beta$  and  $n$ , such that for any  $z_0 \in \partial\mathbb{H}$ ,  $R > 0$  and  $u \in H^1(\mathbb{B}_R(z_0), \mathfrak{w})$  we have*

$$\begin{aligned} & \inf_{c \in \mathbb{R}} \left( \frac{1}{|\mathbb{B}_R(z_0)|_{\beta-1}} \int_{\mathbb{B}_R(z_0)} |u(z) - c|^2 y^{\beta-1} dx dy \right)^{1/2} \\ & \leq C R^2 \left( \frac{1}{|\mathbb{B}_R(z_0)|_{\beta}} \int_{\mathbb{B}_R(z_0)} |\nabla u(z)|^2 y^{\beta} dx dy \right)^{1/2}. \end{aligned} \quad (2.9)$$

To prove Lemma 2.5 and Corollary 2.6, we make use of the following extension property

**Lemma 2.7** (Extension operator). *Let  $z_0 \in \partial\mathbb{H}$  and  $R > 0$ . Let  $D = (a, b) \times (0, c)$  be a rectangle such that  $\mathbb{B}_R(z_0) \subseteq D$ . Then, there exists a continuous extension*

$$E : H^1(\mathbb{B}_R(z_0), \mathfrak{w}) \rightarrow H^1(D, \mathfrak{w}),$$

and there exists a positive constant  $C$ , depending on  $D$ ,  $R$ ,  $n$  and  $\beta$ , such that for any  $u \in H^1(\mathbb{B}_R(z_0), \mathfrak{w})$  we have

$$\begin{aligned} \|Eu\|_{L^2(D, y^{\beta-1})} &\leq C\|u\|_{L^2(\mathbb{B}_R(z_0), y^{\beta-1})}, \\ \|\nabla Eu\|_{L^2(D, y^\beta)} &\leq C\|\nabla u\|_{L^2(\mathbb{B}_R(z_0), y^\beta)}. \end{aligned} \quad (2.10)$$

**Remark 2.8.** Without loss of generality, in the proofs of Lemmas 2.5 & 2.7 and Corollary 2.6 we may assume  $z_0 = (0, 0)$ .

*Proof of Lemma 2.5.* Let  $u \in H^1(\mathbb{B}_R(z_0), \mathfrak{w})$  and choose  $a, b \in \mathbb{R}$  and  $\delta > 0$ , depending only on  $R$ , such that  $\mathbb{B}_R(z_0) \subseteq (a, b) \times (0, \delta)$ . Let  $k > 1$  be such that

$$2k^{-\beta} = \frac{1}{2}, \quad (2.11)$$

and denote by  $D = (a, b) \times (0, k\delta)$ . Let  $\hat{u} = Eu$  be the extension of  $u$  to  $D$  given by Lemma 2.7. Assuming that (2.8) holds for  $\hat{u}$ , we obtain that it holds for  $u$  also in the following way,

$$\begin{aligned} \inf_{c \in \mathbb{R}} \left( \int_{\mathbb{B}_R(z_0)} |u(z) - c|^2 y^{\beta-1} dx dy \right)^{1/2} &\leq \inf_{c \in \mathbb{R}} \left( \int_D |\hat{u}(z) - c|^2 y^{\beta-1} dx dy \right)^{1/2} \\ &\leq C \left( \int_D |\nabla \hat{u}(z)|^2 y^\beta dx dy \right)^{1/2} \\ &\leq C \left( \int_{\mathbb{B}_R(z_0)} |\nabla u(z)|^2 y^\beta dx dy \right)^{1/2}. \end{aligned}$$

In the first and last inequalities above, we made use of (2.10).

Therefore, we may assume  $u \in H^1(D, \mathfrak{w})$ . Our goal is to prove that (2.8) holds for  $u \in H^1(D, \mathfrak{w})$ . By [15, Corollary A.14], we may assume without loss of generality that  $u \in C^1(\bar{D})$ . Let  $c \in \mathbb{R}$  and let  $v = u - c$ . Then, by the mean value theorem, we have for any  $y \in (0, \delta)$  and  $x \in (a, b)$

$$v(x, y) = v(x, ky) + \int_{ky}^y v_y(x, t) dt.$$

Squaring both sides of the preceding equation and integrating in  $y$  with respect to  $y^{\beta-1} dy$ , we obtain

$$\int_0^\delta |v(x, y)|^2 y^{\beta-1} dy \leq 2 \int_0^\delta |v(x, ky)|^2 y^{\beta-1} dy + 2 \int_0^\delta \left| \int_{ky}^y v_y(x, t) dt \right|^2 y^{\beta-1} dy. \quad (2.12)$$

By applying the change of variable  $y' = ky$ , we see that

$$\int_0^\delta |v(x, ky)|^2 y^{\beta-1} dy = k^{-\beta} \int_0^{k\delta} |v(x, y')|^2 y'^{\beta-1} dy'. \quad (2.13)$$

Also, we have for  $\beta \neq 1$ ,

$$\begin{aligned} \int_0^\delta \left| \int_{ky}^y v_y(x, t) dt \right|^2 y^{\beta-1} dy &= \int_0^\delta \left| \int_{ky}^y v_y(x, t) t^{\beta/2} t^{-\beta/2} dt \right|^2 y^{\beta-1} dy \\ &\leq \frac{1}{|1 - \beta|} \int_0^\delta \int_y^{ky} |v_y(x, t)|^2 t^\beta dt \left| y^{-\beta+1} - (ky)^{-\beta+1} \right| y^{\beta-1} dy \\ &\leq \delta \frac{1 + k^{-\beta+1}}{|1 - \beta|} \int_0^{k\delta} |v_y(x, y)|^2 y^\beta dy. \end{aligned} \quad (2.14)$$



For  $\beta = 1$ , we have

$$\begin{aligned} \int_0^\delta \left| \int_{ky}^y v_y(x, t) dt \right|^2 dy &= \int_0^\delta \left| \int_{ky}^y v_y(x, t) t^{1/2} t^{-1/2} dt \right|^2 dy \\ &\leq \int_0^\delta \int_y^{ky} |v_y(x, t)|^2 t dt \log \frac{ky}{y} dy \\ &\leq \delta \log k \int_0^{k\delta} |v_y(x, y)|^2 y dy. \end{aligned} \quad (2.15)$$

Define a positive constant  $C_0 \equiv C_0(\beta, \delta)$  by  $C_0 = 2\delta \frac{1+k^{-\beta+1}}{|1-\beta|}$  when  $\beta \neq 1$ , and  $C_0 = 2\delta \log k$  when  $\beta = 1$ . By combining equations (2.12), (2.13), (2.14) and (2.15), we obtain

$$\begin{aligned} \int_0^\delta |v(x, y)|^2 y^{\beta-1} dy &\leq 2k^{-\beta} \int_0^{k\delta} |v(x, y)|^2 y^{\beta-1} dy + C_0 \int_0^{k\delta} |v_y(x, y)|^2 y^\beta dy \\ &\leq 2k^{-\beta} \int_0^\delta |v(x, y)|^2 y^{\beta-1} dy + 2k^{-\beta} \int_\delta^{k\delta} |v(x, y)|^2 y^{\beta-1} dy \\ &\quad + C_0 \int_0^{k\delta} |v_y(x, y)|^2 y^\beta dy. \end{aligned}$$

Recall that  $k > 1$  was chosen such that (2.11) is satisfied. Therefore, by integrating also in  $x$ , there exists  $C = C(\beta, \delta)$  such that

$$\int_a^b \int_0^{k\delta} |v(x, y)|^2 y^{\beta-1} dy dx \leq C \int_a^b \int_\delta^{k\delta} |v(x, y)|^2 y^{\beta-1} dy dx + C \int_a^b \int_0^{k\delta} |v_y(x, y)|^2 y^\beta dy dx.$$

Since  $v = u - c$ , we have

$$\begin{aligned} &\inf_{c \in \mathbb{R}} \int_D |u(x, y) - c|^2 y^{\beta-1} dy dx \\ &\leq C \inf_{c \in \mathbb{R}} \int_a^b \int_\delta^{k\delta} |u(x, y) - c|^2 y^{\beta-1} dy dx + C \int_D |u_y(x, y)|^2 y^\beta dy dx. \end{aligned}$$

The rectangle  $D' := [a, b] \times [\delta, k\delta]$  is contained in  $\{y > 0\}$ , so the weighted measure  $y^{\beta-1} dy dx$  is equivalent to the Lebesgue measure  $dy dx$ . The rectangle  $D'$  is a convex domain and so we may apply the classical Poincaré inequality [28, Equation (7.45)] to give

$$\inf_{c \in \mathbb{R}} \int_a^b \int_\delta^{k\delta} |u(x, y) - c|^2 y^{\beta-1} dy dx \leq C \int_a^b \int_\delta^{k\delta} |\nabla u(x, y)|^2 y^\beta dy dx.$$

Combining the last two inequalities yields (2.8).  $\square$

**Remark 2.9.** Koch states a weighted Poincaré inequality on the half-space [35, Lemma 4.4.4], with weight  $y^{\beta-1} e^{-\kappa \rho(z, z_0)}$ , where  $\kappa$  is a positive constant,  $z_0$  is a fixed point in  $\mathbb{H}$ , and  $\rho(z, z_0)$  is equivalent to  $d^2(z, z_0)$ , in the sense that there exists a constant  $c > 0$  such that

$$cd^2(z, z_0) \leq \rho(z, z_0) \leq \frac{1}{c} d^2(z, z_0), \forall z \in \mathbb{H}.$$

The proof of this result is long and technical. So, rather than use this result to prove a weighted Poincaré inequality on a ball using an extension principle, we give a much simpler proof for balls and weights  $y^{\beta-1}$  and  $y^\beta$ .



**Remark 2.10.** When  $\beta \geq 1$ , from [15, Lemma A.1 & A.4] we have that  $H_0^1(\mathcal{O}, \mathfrak{w}) = H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ . Then, as in the case of the Poincaré inequality for finite-width domains [1, §6.26], it might be true that the stronger version of (2.8) holds

$$\left( \int_{\mathbb{B}_R(z_0)} |u(z)|^2 y^{\beta-1} dx dy \right)^{1/2} \leq C \left( \int_{\mathbb{B}_R(z_0)} |\nabla u(z)|^2 y^\beta dx dy \right)^{1/2}. \quad (2.16)$$

**Remark 2.11** (Scaling under Koch metric). We have the following scaling property

$$\mathbb{B}_{R_1}(z_0) = \left( \frac{R_1}{R_2} \right)^2 \mathbb{B}_{R_2}(z_0), \quad \forall R_1, R_2 > 0. \quad (2.17)$$

This property follows from the observation that, for any  $z \in \mathbb{H}$ , using the fact that  $z_0 = (0, 0)$ , we have

$$d(z, z_0) = \frac{|z|}{\sqrt{y + |z|}}.$$

Therefore, for any  $z \in \mathbb{H}$ ,

$$d\left(\left(\frac{R_1}{R_2}\right)^2 z, z_0\right) = \frac{R_1}{R_2} d(z, z_0),$$

and so,  $d(z, z_0) < R_2$  if and only if  $d\left((R_1/R_2)^2 z, z_0\right) < R_1$ , from which (2.17) follows.

Notice that (2.17) does not hold if  $z_0 = (y_0, z_0)$  with  $y_0 > 0$ .

*Proof of Corollary 2.9.* Let  $R > 0$  and  $\bar{R} > 0$  and define  $v$  by rescaling

$$u(z) = v\left(\left(\frac{\bar{R}}{R}\right)^2 z\right), \quad \forall z \in \mathbb{B}_R(z_0).$$

The rescaling map defined by

$$z \mapsto \left(\frac{\bar{R}}{R}\right)^2 z,$$

maps  $\mathbb{B}_R(z_0)$  into  $\mathbb{B}_{\bar{R}}(z_0)$  by Remark 2.11. By applying Lemma 2.5 to  $v$  on  $\mathbb{B}_{\bar{R}}(z_0)$ , there is a positive constant  $C$ , depending only on  $\bar{R}$ ,  $n$  and  $\beta$ , such that (2.8) holds. By changing variables, we obtain

$$\inf_{c \in \mathbb{R}} \left(\frac{\bar{R}}{R}\right)^{2(\beta-1)} \int_{\mathbb{B}_R(z_0)} |u - c|^2 y^{\beta-1} dx dy \leq \left(\frac{R}{\bar{R}}\right)^4 \left(\frac{\bar{R}}{R}\right)^{2\beta} \int_{\mathbb{B}_{\bar{R}}(z_0)} |\nabla u|^2 y^\beta dx dy. \quad (2.18)$$

Using Lemma 2.4, we rewrite (2.18) in the following form

$$\inf_{c \in \mathbb{R}} \frac{|\mathbb{B}_{\bar{R}}(z_0)|_{\beta-1}}{|\mathbb{B}_R(z_0)|_{\beta-1}} \int_{\mathbb{B}_R(z_0)} |u - c|^2 y^{\beta-1} dx dy \leq \left(\frac{R}{\bar{R}}\right)^4 \frac{|\mathbb{B}_{\bar{R}}(z_0)|_\beta}{|\mathbb{B}_R(z_0)|_\beta} \int_{\mathbb{B}_{\bar{R}}(z_0)} |\nabla u|^2 y^\beta dx dy,$$

from which (2.9) follows immediately.  $\square$

## 3. JOHN-NIRENBERG INEQUALITY

In this section we recall the abstract John-Nirenberg inequality (Theorem 3.1) due to E. Bombieri and E. Giusti [5] and, in particular, provide a justification — via Proposition 3.2 — that its hypotheses hold in the setting of the problems described in §1.

We restrict the statement of [5, Theorem 4] to the framework of our problems, so in [5, Theorem 4] we choose  $\mathbb{H}$  to be the topological space and  $d\mu = y^{\beta-1} dx dy$  to be the regular positive Borel measure on  $\mathbb{H}$ . Let  $S_r$ ,  $0 \leq r \leq 1$  be a family of non-empty open sets in  $\mathbb{H}$  such that

$$\begin{aligned} S_s &\subseteq S_r, & \forall 0 \leq s \leq r \leq 1, \\ 0 < |S_r|_{\beta-1} &< \infty, & \forall 0 \leq r \leq 1. \end{aligned} \quad (3.1)$$

Let  $w$  be a measurable positive function on  $S_1$ . For  $t \neq 0$  and  $0 \leq r \leq 1$ , we denote by

$$\begin{aligned} |w|_{t,r} &= \left( \frac{1}{|S_r|_{\beta-1}} \int_{S_r} |w|^t y^{\beta-1} dx dy \right)^{1/t}, \\ |w|_{\infty,r} &= \operatorname{ess\,sup}_{S_r} w, \\ |w|_{-\infty,r} &= \operatorname{ess\,inf}_{S_r} w. \end{aligned}$$

We now recall the

**Theorem 3.1** (Abstract John-Nirenberg Inequality). [5, Theorem 4] *Let  $0 < \theta_0, \theta_1 \leq \infty$  and  $w$  be a measurable positive function on  $S_1$  such that*

$$|w|_{\theta_0,1} < \infty \text{ and } |w|_{\theta_1,1} > 0.$$

*Suppose there exist constants  $\gamma > 0$ ,  $0 < t^* \leq \frac{1}{2} \min\{\theta_0, \theta_1\}$  and  $Q > 0$  such that for all  $0 \leq s < r \leq 1$  and  $0 < t \leq t^*$ ,*

$$\begin{aligned} |w|_{\theta_0,s} &\leq [Q(r-s)^\gamma]^{1/\theta_0-1/t} |w|_{t,r}, \\ |w|_{-\theta_1,s} &\geq [Q(r-s)^\gamma]^{1/t-1/\theta_1} |w|_{-t,r}. \end{aligned} \quad (3.2)$$

*Assume further that*

$$A := \sup_{0 \leq r \leq 1} \inf_{c \in \mathbb{R}} \frac{1}{|S_r|_{\beta-1}} \int_{S_r} |\log w - c| y^{\beta-1} dx dy < \infty. \quad (3.3)$$

*Then, we have*

$$|w|_{\theta_0,0} \leq \left( \frac{|S_1|_{\beta-1}}{|S_0|_{\beta-1}} \right)^{1/\theta_0+1/\theta_1} \exp \{ c_2 Q^{-2} (A + 1/t^*) \} |w|_{-\theta_1,1}, \quad (3.4)$$

*where  $c_2$  is a constant depending only on  $\gamma$ , but not on  $Q, \theta_0, \theta_1, t^*, A$  and  $\beta$ .*

In many of our proofs, we will make use of a sequence of cutoff functions,  $\{\eta_N\}_{N \in \mathbb{N}}$ . Let  $\varphi : \mathbb{R} \rightarrow [0, 1]$  be a smooth function such that  $\varphi(x) \equiv 1$  for  $x < 0$ , and  $\varphi \equiv 0$  for  $x > 1$ . Let  $z_0 \in \mathbb{H}$  and let  $\{R_N\}_{N \geq 0}$  be a non-increasing sequence of positive numbers. We define

$$\eta_N(z) := \varphi \left( \frac{1}{R_{N-1}^2 - R_N^2} (d^2(z_0, z) - R_N^2) \right), \quad \forall z \in \bar{\mathbb{H}}, \quad \forall N \in \mathbb{N}. \quad (3.5)$$

Then, the sequence  $\{\eta_N\}_{N \geq 1}$  has the following properties,

$$\eta_N|_{B_{R_N}(z_0)} \equiv 1, \quad \eta_N|_{B_{R_{N-1}}^c(z_0)} \equiv 0, \quad (3.6)$$

$$|\nabla \eta_N| \leq \frac{C}{R_{N-1}^2 - R_N^2}, \quad (3.7)$$

where  $B_{R_{N-1}}^c(z_0) := \mathbb{H} \setminus \bar{B}_{R_{N-1}}(z_0)$  and  $C$  is a positive constant independent of  $N$  and the sequence  $\{R_N\}_{N \geq 0}$ . The bound in (3.7) can be deduced from the calculation,

$$\nabla \eta_N = \varphi' \left( \frac{1}{R_{N-1}^2 - R_N^2} (d^2(z_0, z) - R_N^2) \right) \frac{1}{R_{N-1}^2 - R_N^2} \nabla d^2(z_0, z).$$

Also, we have that  $|\nabla d^2(z_0, z)| \leq 5$ , for all  $z_0, z \in \mathbb{H}$ . Since  $\varphi'$  is also uniformly bounded on  $\mathbb{R}$ , we obtain (3.10).

Similarly, we can construct a sequence of cutoff functions,  $\{\eta_N\}_{N \in \mathbb{N}}$ , when  $\{R_N\}_{N \geq 0}$  is a non-decreasing sequence of positive numbers.

We now provide a justification that the hypotheses of Theorem 3.1 hold in the setting of the problems discussed in this article.

**Proposition 3.2** (Application of Theorem 3.1). *Let  $z_0 \in \partial \mathbb{H}$  and  $0 < 4R \leq 1$ . Let  $S_r = \mathbb{B}_{(2+r)R}(z_0)$ , for all  $0 \leq r \leq 1$ . Let  $\theta_0, \theta_1$  be as in Theorem 3.1 and set  $t^* = \frac{1}{2} \min\{\theta_0, \theta_1\}$ . Then, there exist positive constants  $Q$  and  $\gamma$ , independent of  $R$  and  $z_0$ , such that (3.4) holds for any bounded positive function  $w$  on  $S_1$  which satisfies the energy estimate (5.12) or (7.3).*

*Proof.* We begin by proving the first inequality in (3.2) by applying Moser iteration finitely many times. The second inequality in (3.2) can be proved by a similar technique. We outline the proof when  $w$  satisfies the energy estimate (5.12), but the proof applies as well to positive bounded functions  $w$  satisfying the energy estimate (7.3).

First, we consider the *special case* when  $\theta_0$  and  $t$  satisfy the requirement: There exists an integer  $N^* \geq 1$  such that  $\theta_0$  can be written as

$$\theta_0 = t \left( \frac{p}{2} \right)^{N^*}. \quad (3.8)$$

Let  $0 \leq s < r \leq 1$  and set  $R_0 = (2+r)R$ . We denote

$$c := \sum_{k=1}^{\infty} \frac{1}{k^2}$$

and we let

$$R_N^2 := \left( (2+r)^2 - (r-s)^2 \sum_{k=1}^N \frac{1}{ck^2} \right) R^2, \quad \forall N = 1, \dots, N^*. \quad (3.9)$$

We observe that  $(2+s)R < R_N < R_{N-1} \leq (2+r)R$ . Let  $\{\eta_N\}_{N \in \mathbb{N}}$  be a sequence of non-negative, smooth cutoff functions as constructed in (3.5), by choosing  $R_N$  as in (3.9). Then, (3.7) becomes

$$|\nabla \eta_N| \leq \frac{CN^2}{R^2(r-s)^2}. \quad (3.10)$$

Let  $P_N := t(p/2)^N$ , for  $N = 1, \dots, N^*$ , and  $\alpha_N = p_N - 1$ , for all  $N = 0, \dots, N^* - 1$ . We set

$$I(N) := \left( \int_{\mathbb{B}_{R_N}(z_0)} |w|^{p_N} y^{\beta-1} dx dy \right)^{1/p_N}, \quad (3.11)$$

From our hypothesis,  $w$  satisfies (5.12), that is,

$$\|\eta w^{(\alpha+1)/2}\|_{L^p(\mathbb{H}, y^{\beta-1})} \leq C_0(R, \alpha) \|w^{(\alpha+1)/2}\|_{L^2(\text{supp } \eta, y^{\beta-1})}, \quad (3.12)$$

where

$$C_0(R, \alpha) := [C|1 + \alpha|]^{(\xi+1)/p} \left(1 + \|\sqrt{y}\nabla\eta\|_{L^\infty(\mathbb{H})}^2\right)^{1/p}, \quad (3.13)$$

and  $\xi$  and  $C$  are positive constants, independent of  $w$ ,  $\alpha$  and  $\eta$ . We choose  $\alpha = \alpha_{N-1}$  and  $\eta = \eta_N$  in (3.12), so the definition (3.11) gives us, for all  $N \geq 1$ ,

$$I(N) \leq C_1(R, r, s, N) I(N-1), \quad (3.14)$$

where

$$C_1(R, r, s, N) := (C|p_{N-1}|)^{(\xi+1)/p_N} \left(1 + \|\sqrt{y}\nabla\eta_N\|_{L^\infty(\mathbb{H})}^2\right)^{1/p_N}.$$

From Lemma 2.4, we have  $y \leq CR^2$  on  $\mathbb{B}_{R_N}(z_0)$ , where  $C$  is a positive constant independent of  $R$  and  $N$ . Using the bound (3.10), we obtain

$$C_1(R, r, s, N) := (C|p_{N-1}|)^{(\xi+1)/p_N} \left(\frac{CN^4}{R^2(r-s)^4}\right)^{1/p_N}.$$

By iterating inequality (3.14), we obtain

$$I(N^*) \leq C_2(R, r, s) I(0), \quad (3.15)$$

where

$$C_2(R, r, s) := \prod_{N=1}^{N^*} \left[ Cp_{N-1}^{\xi+1} N^4 R^{-2} (r-s)^{-4} \right]^{1/p_N}. \quad (3.16)$$

Next, we prove the

**Claim 3.3.** *There are positive constants  $Q$  and  $\gamma$ , independent of  $N^*$ ,  $R$ ,  $r$  and  $s$ , such that*

$$C_2(R, r, s) \leq (Q(r-s)^\gamma)^{1/\theta_0-1/t} R^{\frac{4}{p-2}(1/\theta_0-1/t)}. \quad (3.17)$$

*Proof of Claim 3.3.* We can rewrite the expression (3.16) for  $C_2(R, r, s)$  to obtain

$$\begin{aligned} C_2(R, r, s) &= \prod_{N=1}^{N^*} \left[ Ct^{\xi+1} R^{-2} (r-s)^{-4} \right]^{1/p_N} \left[ \left(\frac{p}{2}\right)^{N-1} N^4 \right]^{1/p_N} \\ &\leq \left[ Ct^{\xi+1} R^{-2} (r-s)^{-4} \right]^{\sum_{N=1}^{N^*} 1/p_N} \left( C \frac{p}{2} \right)^{\sum_{N=1}^{N^*} N/p_N}, \end{aligned}$$

where we used in the last line that  $N^4 \leq C(p/2)^N$ , for some positive constant  $C$  depending only on  $p$ . Thus,

$$C_2(R, r, s) \leq \left[ Ct^{\xi+1} R^{-2} (r-s)^{-4} \right]^{\sum_{N=1}^{N^*} 1/p_N} \left( C \frac{p}{2} \right)^{\sum_{N=1}^{N^*} N/p_N}. \quad (3.18)$$

Recall that

$$\sum_{N=1}^{N^*} x^N = x \frac{1-x^{N^*}}{1-x} \quad \text{and} \quad \sum_{N=1}^{N^*} Nx^N = x^2 \frac{1-x^{N^*}}{1-x}.$$

Hence, (3.8) leads to the identities

$$\sum_{N=1}^{N^*} \frac{1}{p_N} = \frac{2}{p-2} \left( \frac{1}{t} - \frac{1}{\theta_0} \right) \quad \text{and} \quad \sum_{N=1}^{N^*} \frac{N}{p_N} = \frac{4}{p(p-2)} \left( \frac{1}{t} - \frac{1}{\theta_0} \right).$$

Therefore, inequality (3.17) becomes

$$C_2(R, r, s) \leq [R^{-2}(r-s)^{-4}]^{\frac{2}{p-2}(\frac{1}{t}-\frac{1}{\theta_0})} \left( C\theta_0^{\xi+1} \frac{p}{2} \right)^{\frac{4}{p(p-2)}(\frac{1}{t}-\frac{1}{\theta_0})}, \quad (3.19)$$

which is equivalent to (3.17) with the choice of the constants  $Q = \left( C\theta_0^{\xi+1} p/2 \right)^{-1}$  and  $\gamma = 8/(p-2)$ . This completes the proof of Claim 3.3.  $\square$

Using the fact that  $4/(p-2) = 2(n+\beta-1)$ , Lemma 2.4 yields

$$\frac{|\mathbb{B}_{(2+s)R}(z_0)|_{\beta-1}^{1/\theta_0}}{|\mathbb{B}_{(2+r)R}(z_0)|_{\beta-1}^{1/t}} \geq C^{1/\theta_0+1/t} R^{4/(p-2)(1/\theta_0-1/t)},$$

for some positive constant  $C < 1$ . Therefore, inequality (3.19) becomes

$$C_2(R, r, s) \leq C^{-1/\theta_0-1/t} (Q(r-s)^\gamma)^{1/\theta_0-1/t} \frac{|\mathbb{B}_{(2+s)R}(z_0)|_{\beta-1}^{1/\theta_0}}{|\mathbb{B}_{(2+r)R}(z_0)|_{\beta-1}^{1/t}}. \quad (3.20)$$

From our hypothesis,  $t \leq t^* \leq \theta_0/2$ , we have

$$3(1/\theta_0 - 1/t) \leq -1/\theta_0 - 1/t \leq 1/\theta_0 - 1/t,$$

and so, for a new positive constant  $Q$ , the inequality (3.20) leads to

$$C_2(R, r, s) \leq (Q(r-s)^\gamma)^{1/\theta_0-1/t} \frac{|\mathbb{B}_{(2+s)R}(z_0)|_{\beta-1}^{1/\theta_0}}{|\mathbb{B}_{(2+r)R}(z_0)|_{\beta-1}^{1/t}}. \quad (3.21)$$

By employing the inequalities (3.21) and (3.15) and the definition (3.11) of  $I(N)$ , we obtain

$$\begin{aligned} & \left( \int_{\mathbb{B}_{(2+s)R}(z_0)} |w|^{\theta_0} y^{\beta-1} dx dy \right)^{1/\theta_0} \leq I(N^*) \\ & \leq (Q(r-s)^\gamma)^{1/\theta_0-1/t} \frac{|\mathbb{B}_{(2+s)R}(z_0)|_{\beta-1}^{1/\theta_0}}{|\mathbb{B}_{(2+r)R}(z_0)|_{\beta-1}^{1/t}} I(0) \\ & = (Q(r-s)^\gamma)^{1/\theta_0-1/t} \frac{|\mathbb{B}_{(2+s)R}(z_0)|_{\beta-1}^{1/\theta_0}}{|\mathbb{B}_{(2+r)R}(z_0)|_{\beta-1}^{1/t}} \left( \int_{\mathbb{B}_{(2+r)R}(z_0)} |w|^t y^{\beta-1} dx dy \right)^{1/t}, \end{aligned}$$

from which we readily obtain the first inequality in (3.2), in the *special case* where  $t$  and  $\theta_0$  satisfy (3.8) for some integer  $N^* \geq 1$ .

Next, we show that the first inequality in (3.2) holds for *any*  $t \in (0, t^*)$ . For this purpose, we choose an integer  $N^* \geq 1$  such that

$$t \left( \frac{p}{2} \right)^{N^*-1} < \theta_0 < t \left( \frac{p}{2} \right)^{N^*}.$$

We denote  $\theta_0^* = t(p/2)^{N^*}$  and we apply the previous analysis to  $t$  and  $\theta_0^*$ , which now satisfy (3.8), to give

$$|w|_{\theta_0^*, s} \leq (Q(r-s)^\gamma)^{1/\theta_0^*-1/t} |w|_{t, r}.$$

Using Hölder's inequality with  $p = \theta_0^*/\theta_0 > 1$ , we find that

$$|w|_{\theta_0, s} \leq |w|_{\theta_0^*, s},$$

and so

$$\begin{aligned} |w|_{\theta_0, s} &\leq (Q(r-s)^\gamma)^{1/\theta_0^*-1/t} |w|_{t, r} \\ &\leq (Q(r-s)^\gamma)^{\frac{1/\theta_0^*-1/t}{1/\theta_0-1/t}} |w|_{t, r}. \end{aligned}$$

Notice that  $2\theta_0^*/p \leq \theta_0 \leq \theta_0^*$  and  $0 < t < \theta_0/2$ . Then,

$$1 \leq \frac{1/\theta_0^* - 1/t}{1/\theta_0 - 1/t} \leq \frac{1/\theta_0^* - 1/t}{p/2\theta_0^* - 1/t} \leq \frac{(2/p)^{N^*} - 1}{(2/p)^{N^*+1} - 1} \leq \frac{p}{p-2}.$$

Consequently, we define  $\tilde{Q}$  to be  $Q^{p/(p-2)}$  if  $Q < 1$ , and we leave  $Q$  unchanged if  $Q \geq 1$  and, setting  $\tilde{\gamma} := \gamma p/(p-2)$ , the preceding estimate for  $|w|_{\theta_0, s}$  becomes

$$|w|_{\theta_0, s} \leq \left( \tilde{Q}(r-s)^{\tilde{\gamma}} \right)^{1/\theta_0-1/t} |w|_{t, r},$$

which is precisely the first inequality in (3.2).  $\square$

#### 4. SUPREMUM ESTIMATES NEAR THE PORTION OF THE BOUNDARY WHERE $A$ IS DEGENERATE

In this section, we prove Theorem 1.7, that is, local boundedness up to  $\bar{\Gamma}_0$  for solutions,  $u$ , to the variational equation (1.18). Our choice of test functions when applying Moser iteration follows that employed in the proof of [28, Theorem 8.15]. However, the choice of test functions used in the proof of the classical local supremum estimates [28, Theorem 8.17] is not suitable in our case because the test functions in (1.18) are not required to satisfy a homogenous Dirichlet boundary condition along  $\bar{\Gamma}_0$ . In addition, the method of deriving the energy estimate (4.8) is slightly different from [28, Theorem 8.18] because, instead of using the classical Sobolev inequalities [28, Theorem 7.10], we use Lemma 2.2.

We start first with the following observation

**Lemma 4.1.** *Let  $K$  be a finite, right circular cone and  $\mathcal{O}$  be a domain which obeys the uniform interior and exterior cone condition on  $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$  with cone  $K$ . Then, there are positive constants  $\bar{R}$  and  $c$  depending on  $K$ ,  $n$  and  $\beta$  such that, for all  $R \in (0, \bar{R}]$ , we have*

$$c^{-1} |\mathbb{B}_R(z_0)|_{\beta-1} \leq |B_R(z_0)|_{\beta-1} \leq c |\mathbb{B}_R(z_0)|_{\beta-1}, \quad \forall z_0 \in \bar{\Gamma}_0, \quad (4.1)$$

and also

$$c^{-1} |\mathbb{B}_R(z_0)|_{\beta-1} \leq |\mathbb{B}_R(z_0) \setminus B_R(z_0)|_{\beta-1} \leq c |\mathbb{B}_R(z_0)|_{\beta-1}, \quad \forall z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1. \quad (4.2)$$

Example A.1 illustrates a domain  $\mathcal{O}$  which does not satisfy condition (4.1).

*Proof of Lemma 4.1.* We only outline the proof of inequality (4.1) because (4.2) follows similarly. Notice that the second inequality in (4.1) holds trivially, for any  $c \geq 1$ , because  $B_R(z_0) \subseteq \mathbb{B}_R(z_0)$ , for all  $R \geq 0$ .

Because the domain  $\mathcal{O}$  obeys the uniform interior and exterior cone condition in Definition 1.4, we can find positive constants  $\hat{R}$  and  $\hat{c}$  such that, for all  $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$ , one of the following two conditions hold

$$\begin{aligned} \{z \in \mathbb{E}_{\hat{R}}(z_0) : 0 < y < \hat{c}(x - x_0)\} &\subseteq \mathcal{O} \\ \text{and } \{z \in \mathbb{E}_{\hat{R}}(z_0) : 0 < y < \hat{c}(x_0 - x)\} &\subseteq \mathbb{H} \setminus \mathcal{O}, \end{aligned} \quad (4.3)$$

or

$$\begin{aligned} \{z \in \mathbb{E}_{\hat{R}}(z_0) : 0 < y < \hat{c}(x_0 - x)\} &\subset \mathcal{O} \\ \text{and } \{z \in \mathbb{E}_{\hat{R}}(z_0) : 0 < y < \hat{c}(x - x_0)\} &\subset \mathbb{H} \setminus \mathcal{O}. \end{aligned} \quad (4.4)$$

We fix a point  $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$  and, without loss of generality, we may assume that  $z_0 = (0, 0)$  and condition (4.3) is satisfied. The second inclusion in (2.7) in Lemma 2.4 gives us that by setting  $\bar{R}_1 := \sqrt{\hat{R}}$ , we have

$$\mathbb{B}_{\bar{R}_1}(z_0) \subseteq \mathbb{E}_{\hat{R}}(z_0),$$

and so by condition (4.3) we have

$$\{z \in \mathbb{B}_{\bar{R}_1}(z_0) : 0 < y < \hat{c}x\} \subseteq \mathcal{O}.$$

We fix  $R \in (0, \bar{R}_1]$ , and we denote for simplicity

$$S := \{z \in \mathbb{B}_R(z_0) : 0 < y < \hat{c}x\}.$$

Then, obviously we have  $S \subseteq B_R(z_0)$ . As in Lemma 2.4, we denote by

$$R_1 := R^2/2000, \tag{4.5}$$

and we let

$$S_1 := \{z \in \mathbb{E}_{R_1}(z_0) : 0 < y < \hat{c}x\}.$$

By (2.7), it follows that

$$S_1 \subseteq S \subseteq B_R(z_0).$$

By direct calculations, we obtain the following inequality for  $|S_1|_{\beta-1}$ ,

$$\begin{aligned} |S_1|_{\beta-1} &\geq \int_0^{R_1/\sqrt{\hat{c}^2+1}} \int_0^{\hat{c}x} y^{\beta-1} dy dx \\ &= \frac{\hat{c}^\beta}{\beta(\beta+1)} \frac{R_1^{\beta+1}}{(\hat{c}^2+1)^{(\beta+1)/2}}. \end{aligned}$$

Identity (4.5) and the preceding inequality show that we can find a large enough constant  $c \geq 1$ , depending only on  $\beta$ ,  $\hat{c}$  and  $n$ , such that

$$c^{-1}R^{2(\beta+1)} \leq |S_1|_{\beta-1} \leq |B_R(z_0)|_{\beta-1} \leq cR^{2(\beta+1)}, \quad \forall 0 < R \leq \bar{R}_1.$$

Conclusion (4.1) for points  $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$  now follows from the preceding inequalities and (2.6).

We consider now the case of points  $z_0 \in \Gamma_0$ . If the Euclidean distance between  $z_0$  and  $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$  is greater than  $\hat{R}/2$ , then  $\mathbb{E}_{\hat{R}/2}(z_0) \subset \mathcal{O}$  and the second inclusion in (2.7) shows that  $B_R(z_0) = \mathbb{B}_R(z_0)$ , for all  $0 < R \leq \bar{R}_1/\sqrt{2}$ . Thus, inequality (4.1) follows immediately, for all  $0 < R < \bar{R}/\sqrt{2}$ . Otherwise, we can find a point  $z'_0 = (x'_0, 0) \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$ , such that  $|z_0 - z'_0| < \hat{R}/2$ . Without loss of generality, we may assume that the inclusion condition (4.3) is satisfied at  $z'_0$ , which gives us that

$$z_0 \in \{z \in \mathbb{E}_{\hat{R}}(z'_0) : 0 < y < \hat{c}(x - x'_0)\} \subset \mathcal{O},$$

and also, since  $|z_0 - z'_0| < \hat{R}/2$ ,

$$\{z \in \mathbb{E}_{\hat{R}/2}(z_0) : 0 < y < \hat{c}(x - x_0)\} \subset \mathcal{O}.$$

We may now apply the same argument used for points  $z'_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$  to points  $z_0 \in \Gamma_0$  satisfying the preceding inclusion condition, to conclude that (4.1) holds, for all  $0 < R \leq \bar{R}_1\sqrt{2}$ .

By combining the preceding cases we find that (4.1) holds, for all  $0 < R \leq \bar{R}_1\sqrt{2}$ , so we choose  $\bar{R} := \bar{R}_1\sqrt{2}$ .  $\square$

*Proof of Theorem 1.7.* Let  $\bar{R}$  be as in Lemma 4.1. We organize the proof in several steps.



**Step 1** (Energy estimates). Let  $\alpha \geq 1$  and let  $\eta \in C_0^1(\bar{\mathbb{H}})$  be a non-negative cutoff function with support in  $\bar{\mathbb{B}}_{2R}(z_0)$ . We define

$$A := \|f\|_{L^s(\text{supp } \eta, y^{\beta-1})}. \quad (4.6)$$

We will apply the following calculations in Steps 1 and 2 to two choices of  $w$ , namely,

$$w := u^+ + A \text{ and } w := u^- + A. \quad (4.7)$$

For concreteness, we will illustrate our calculations with the choice

$$w = u^+ + A,$$

but they apply equally well to the choice  $w = u^- + A$ . Our goal is to prove the following

**Claim 4.2** (Energy estimate). *There is a positive constant  $C$ , depending only on the coefficients of the Heston operator (1.4),  $n$ ,  $s$  and  $\bar{R}$ , and there is a positive constant  $\xi$ , depending only on  $n$ ,  $\beta$  and  $s$ , such that*

$$\begin{aligned} & \left( \int_{\mathcal{O}} |\eta w^\alpha|^p y^{\beta-1} dx dy \right)^{1/p} \\ & \leq (C\alpha)^{\xi+1} \left( \|\sqrt{y} \nabla \eta\|_{L^\infty(\mathbb{H})}^{2/p} + |\text{supp } \eta|_{\beta-1}^{1/p-1/2} \right) \left( \int_{\text{supp } \eta} w^{2\alpha} y^{\beta-1} dx dy \right)^{1/2}. \end{aligned} \quad (4.8)$$

*Proof of Claim 4.2.* We fix  $k \in \mathbb{N}$ . Similarly to the proof of [28, Theorem 8.15], we consider the functions  $H_k : \mathbb{R} \rightarrow [0, \infty)$ ,

$$H_k(t) := \begin{cases} 0, & t < A, \\ t^\alpha - A^\alpha, & A \leq t \leq k, \\ \alpha k^{\alpha-1}(t - k) + H_k(k), & t > k. \end{cases} \quad (4.9)$$

and

$$G_k(t) = \int_0^t |H'_k(s)|^2 ds. \quad (4.10)$$

Then,

$$v = G_k(w)\eta^2 \quad (4.11)$$

is a valid test function in  $H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  in (1.14), by Lemma A.2. Because  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  obeys (1.18) for all  $v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  with support in  $\bar{B}_{2R}(z_0)$ , then the expression (1.14) for  $a(u, v)$  yields

$$\begin{aligned} 0 &= a(u, v) - (f, v)_{L^2(\mathcal{O}, m)} \\ &= \frac{1}{2} \int_{\mathcal{O}} (u_x v_x + \rho \sigma u_x v_y + \rho \sigma u_y v_x + \sigma^2 u_y v_y) y \mathfrak{w} dx dy \\ &\quad - \int_{\mathcal{O}} \left( a_1 u_x + \frac{\gamma}{2} (u_x + \rho \sigma u_y) \text{sign}(x) \right) v y \mathfrak{w} dx dy + \int_{\mathcal{O}} (ru - f) v \mathfrak{w} dx dy. \end{aligned}$$

Since  $\nabla v = G'_k(w)\eta^2\nabla w + 2G_k(w)\eta\nabla\eta$  and the fact that  $G_k(w) = 0$  when  $w \leq A$ , that is,  $u^+ = 0$ , the preceding identity becomes

$$\begin{aligned} & \frac{1}{2} \int_{\mathcal{O}} (w_x^2 + 2\rho\sigma w_x w_y + \sigma^2 w_y^2) G'_k(w) \eta^2 y \mathfrak{w} \, dx \, dy \\ &= - \int_{\mathcal{O}} (w_x \eta_x + \rho\sigma w_x \eta_y + \rho\sigma w_y \eta_x + \sigma^2 w_y \eta_y) G_k(w) \eta y \mathfrak{w} \, dx \, dy \\ & \quad + \int_{\mathcal{O}} \left[ a_1 w_x + \frac{\gamma}{2} (w_x + \rho\sigma w_y) \operatorname{sign}(x) \right] G_k(w) \eta^2 y \mathfrak{w} \, dx \, dy \\ & \quad - \int_{\mathcal{O}} (ru^+ - f) G_k(w) \eta^2 \mathfrak{w} \, dx \, dy. \end{aligned}$$

For convenience, we write the identity as  $I_1 = I_2 + I_3 + I_4$ . From the uniform ellipticity (1.16), we obtain for  $I_1$  that

$$C \int_{\mathcal{O}} |\nabla w|^2 \eta^2 G'_k(w) y \mathfrak{w} \, dx \, dy \leq I_1,$$

where  $C$  is a positive constant depending only on the coefficients of the Heston operator. We notice that  $0 \leq G_k(w) \leq w G'_k(w)$  because  $G'_k(w) = |H'_k(w)|^2$  is a non-decreasing function. Using this fact and that  $w \geq A$ , we obtain for the integrals  $I_2, I_3, I_4$  that there exists a positive constant  $C$ , depending only on the coefficients of the Heston operator, such that

$$\begin{aligned} |I_2| &\leq \frac{1}{2} \int_{\mathcal{O}} (|w_x \eta| |w \eta_x| + \rho\sigma |w_x \eta| |w \eta_y| + \rho\sigma |w_y \eta| |w \eta_x| + \sigma^2 |w_y \eta| |w \eta_y|) G'_k(w) y \mathfrak{w} \, dx \, dy \\ &\leq \varepsilon \int_{\mathcal{O}} |\nabla w|^2 \eta^2 G'_k(w) y \mathfrak{w} \, dx \, dy + \frac{C}{\varepsilon} \int_{\mathcal{O}} |w|^2 |\nabla \eta|^2 G'_k(w) y \mathfrak{w} \, dx \, dy, \\ |I_3| &\leq \varepsilon \int_{\mathcal{O}} |\nabla w|^2 \eta^2 G'_k(w) y \mathfrak{w} \, dx \, dy + \frac{C}{\varepsilon} \int_{\mathcal{O}} |w|^2 |\eta|^2 G'_k(w) y \mathfrak{w} \, dx \, dy, \\ |I_4| &\leq r \int_{\mathcal{O}} w^2 G'_k(w) \eta^2 \mathfrak{w} \, dx \, dy + \int_{\mathcal{O}} |f| w G'_k(w) \eta^2 \mathfrak{w} \, dx \, dy \\ &\leq C \int_{\mathcal{O}} \left( 1 + \frac{|f|}{A} \right) w^2 G'_k(w) \eta^2 \mathfrak{w} \, dx \, dy, \end{aligned}$$

where  $\varepsilon > 0$ . Choosing  $\varepsilon$  small enough, we obtain for a positive constant  $C$ , depending on the coefficients of the Heston operator and  $\bar{R}$ , that

$$\begin{aligned} \int_{\mathcal{O}} |\nabla w|^2 \eta^2 G'_k(w) y^\beta \, dx \, dy &\leq C \left[ \int_{\mathcal{O}} \eta^2 \frac{|f|}{A} w^2 G'_k(w) y^{\beta-1} \, dx \, dy \right. \\ &\quad \left. + \int_{\mathcal{O}} (\eta^2 + y |\nabla \eta|^2) w^2 G'_k(w) y^{\beta-1} \, dx \, dy \right]. \end{aligned} \tag{4.12}$$

Hölder's inequality applied to the conjugate pair  $(s, s^*)$  gives

$$\begin{aligned} & \int_{\mathcal{O}} \eta^2 \frac{|f|}{A} w^2 G'_k(w) y^{\beta-1} \, dx \, dy \\ & \leq \left( \int_{\operatorname{supp} \eta} \frac{|f|^s}{A^s} y^{\beta-1} \, dx \, dy \right)^{1/s} \left( \int_{\mathcal{O}} |\eta^2 w^2 G'_k(w)|^{s^*} y^{\beta-1} \, dx \, dy \right)^{1/s^*}, \end{aligned}$$

and thus, by definition (4.6) of  $A$ ,

$$\int_{\mathcal{O}} \eta^2 \frac{|f|}{A} w^2 G'_k(w) y^{\beta-1} dx dy \leq \left( \int_{\mathcal{O}} |\eta^2 w^2 G'_k(w)|^{s^*} y^{\beta-1} dx dy \right)^{1/s^*}. \quad (4.13)$$

We need to justify first that the right hand side in (4.13) is finite. First, we notice that the following identities hold

$$\begin{aligned} |\nabla H_k(w)|^2 &= |\nabla w|^2 |H'_k(w)|^2 = |\nabla w|^2 G'_k(w), \\ |w H'_k(w)|^2 &= |w|^2 G'_k(w), \end{aligned} \quad (4.14)$$

From the hypothesis  $s > n + \beta$  in Theorem 1.7, we observe that  $2 < 2s^* < p$ , so we may apply the interpolation inequality [28, Inequality (7.10)]. For any  $\varepsilon \in (0, 1)$ , we have

$$\|\eta w H'_k(w)\|_{L^{2s^*}(\mathbb{H}, y^{\beta-1})} \leq \varepsilon \|\eta w H'_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})} + \varepsilon^{-\xi} \|\eta w H'_k(w)\|_{L^2(\mathbb{H}, y^{\beta-1})}, \quad (4.15)$$

where

$$\xi \equiv \xi(p, s) := \frac{p(s^* - 1)}{p - 2s^*}. \quad (4.16)$$

We notice that  $|H'_k(w)| \leq \alpha k^{\alpha-1}$  and  $\eta w \in H^1(\mathcal{O}, \mathfrak{w})$  has compact support in  $\bar{B}_{2R}(z_0)$ . Therefore, we may apply Lemma 2.7 to build an extension  $\hat{w}$  of  $\eta w$  to a rectangle  $D$  containing  $\bar{B}_{2R}(z_0)$ . Lemma 2.2, shows that  $\hat{w} \in L^p(D, y^{\beta-1})$ , which implies that

$$\|\eta w H'_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})} < \infty,$$

and so, the right hand side of (4.13) is finite.

Inequalities (4.12) and (4.13), together with the identities (4.14) yield

$$\begin{aligned} \int_{\mathcal{O}} \eta^2 |\nabla H_k(w)|^2 y^{\beta} dx dy &\leq C \left[ \left( \int_{\mathcal{O}} |\eta w H'_k(w)|^{2s^*} y^{\beta-1} dx dy \right)^{1/s^*} \right. \\ &\quad \left. + \int_{\mathcal{O}} (\eta^2 + y |\nabla \eta|^2) |w H'_k(w)|^2 y^{\beta-1} dx dy \right]. \end{aligned} \quad (4.17)$$

From Lemma 2.2, we obtain

$$\begin{aligned} \int_{\mathcal{O}} |\eta H_k(w)|^p y^{\beta-1} dx dy &\leq \left( \int_{\mathcal{O}} \eta^2 |H_k(w)|^2 y^{\beta-1} dx dy \right)^{(p-2)/2} \int_{\mathcal{O}} |\nabla(\eta H_k(w))|^2 y^{\beta} dx dy \\ &\leq 2 \left( \int_{\mathcal{O}} \eta^2 |H_k(w)|^2 y^{\beta-1} dx dy \right)^{(p-2)/2} \\ &\quad \times \left( \int_{\mathcal{O}} |\nabla \eta|^2 |H_k(w)|^2 y^{\beta} dx dy + \int_{\mathcal{O}} \eta^2 |\nabla H_k(w)|^2 y^{\beta} dx dy \right). \end{aligned} \quad (4.18)$$

Using  $H_k(w) \leq w H'_k(w)$  and inequality (4.17) in (4.18), we see that

$$\begin{aligned} \int_{\mathcal{O}} |\eta H_k(w)|^p y^{\beta-1} dx dy &\leq C \left[ \left( 1 + \|\sqrt{y} \nabla \eta\|_{L^\infty(\mathbb{H})}^2 \right) \left( \int_{\text{supp } \eta} |w H'_k(w)|^2 y^{\beta-1} dx dy \right)^{p/2} \right. \\ &\quad \left. + \left( \int_{\mathcal{O}} |\eta w H'_k(w)|^2 y^{\beta-1} dx dy \right)^{(p-2)/2} \left( \int_{\mathcal{O}} |\eta w H'_k(w)|^{2s^*} y^{\beta-1} dx dy \right)^{1/s^*} \right], \end{aligned} \quad (4.19)$$

where  $C$  is a positive constant depending on the coefficients of the Heston operator and  $\bar{R}$ . We rewrite the estimate for  $\eta w H'_k(w)$  in (4.15) in the form

$$\begin{aligned} \left( \int_{\mathcal{O}} |\eta w H'_k(w)|^{2s^*} y^{\beta-1} dx dy \right)^{1/s^*} &= \|\eta w H'_k(w)\|_{L^{2s^*}(\mathbb{H}, y^{\beta-1})}^2 \\ &\leq 2\varepsilon^2 \|\eta w H'_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^2 + 2\varepsilon^{-2\xi} \|\eta w H'_k(w)\|_{L^2(\mathbb{H}, y^{\beta-1})}^2. \end{aligned}$$

Applying the preceding inequality in (4.19), we obtain

$$\begin{aligned} \|\eta H_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^p &\leq C \left( 1 + \|\sqrt{y} \nabla \eta\|_{L^\infty(\mathbb{H})}^2 \right) \|w H'_k(w)\|_{L^2(\text{supp } \eta, y^{\beta-1})}^p \\ &\quad + C \|\eta w H'_k(w)\|_{L^2(\text{supp } \eta, y^{\beta-1})}^{p-2} \left( \varepsilon^2 \|\eta w H'_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^2 + \varepsilon^{-2\xi} \|\eta w H'_k(w)\|_{L^2(\mathbb{H}, y^{\beta-1})}^2 \right). \end{aligned}$$

By recombining terms in the preceding inequality, we find that

$$\begin{aligned} \|\eta H_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^p &\leq C(1 + \varepsilon^{-2\xi}) \left( 1 + \|\sqrt{y} \nabla \eta\|_{L^\infty(\mathbb{H})}^2 \right) \|w H'_k(w)\|_{L^2(\text{supp } \eta, y^{\beta-1})}^p \\ &\quad + C\varepsilon^2 \|\eta w H'_k(w)\|_{L^2(\mathbb{H}, y^{\beta-1})}^{p-2} \|\eta w H'_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^2. \end{aligned}$$

To estimate the last term in the preceding inequality, we apply Young's inequality with the conjugate pair of exponents,  $(p/2, p/(p-2))$ , to give

$$\begin{aligned} \|\eta w H'_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^2 \|\eta w H'_k(w)\|_{L^2(\mathbb{H}, y^{\beta-1})}^{p-2} \\ \leq \frac{2}{p} \|\eta w H'_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^p + \frac{p-2}{p} \|\eta w H'_k(w)\|_{L^2(\mathbb{H}, y^{\beta-1})}^p. \end{aligned}$$

Combining the previous two inequalities yields

$$\begin{aligned} \|\eta H_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^p &\leq C \left( 1 + (\varepsilon^2 + \varepsilon^{-2\xi}) \right) \left( 1 + \|\sqrt{y} \nabla \eta\|_{L^\infty(\mathbb{H})}^2 \right) \|w H'_k(w)\|_{L^2(\text{supp } \eta, y^{\beta-1})}^p \\ &\quad + C\varepsilon^2 \|\eta w H'_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^p, \end{aligned} \tag{4.20}$$

Employing the definition (4.9) of  $H_k(w)$  gives  $0 \leq w H'_k(w) \leq \alpha H_k(w) + \alpha A^\alpha$ , and so

$$\begin{aligned} \int_{\mathcal{O}} |\eta w H'_k(w)|^p y^{\beta-1} dx dy &\leq |2\alpha|^p \left[ \int_{\mathcal{O}} |\eta H_k(w)|^p y^{\beta-1} dx dy + \int_{\mathcal{O}} |\eta A^\alpha|^p y^{\beta-1} dx dy \right] \\ &\leq |2\alpha|^p \left[ \int_{\mathcal{O}} |\eta H_k(w)|^p y^{\beta-1} dx dy + |\text{supp } \eta|_{\beta-1} A^{\alpha p} \right], \end{aligned}$$

and thus, applying inequality (4.20) yields

$$\begin{aligned} \int_{\mathcal{O}} |\eta H_k(w)|^p y^{\beta-1} dx dy &\leq C \left( 1 + (\varepsilon^2 + \varepsilon^{-2\xi}) \right) \left( 1 + \|\sqrt{y} \nabla \eta\|_{L^\infty(\mathbb{H})}^2 \right) \|w H'_k(w)\|_{L^2(\text{supp } \eta, y^{\beta-1})}^p \\ &\quad + C|2\alpha|^p \varepsilon^2 \left( \|\eta H_k(w)\|_{L^p(y^{\mathbb{H}, \beta-1})}^p + |\text{supp } \eta|_{\beta-1} A^{\alpha p} \right). \end{aligned}$$

By choosing  $\varepsilon = 1/(2\sqrt{C(2\alpha)^p})$  and taking  $p$ -th order roots, we obtain

$$\begin{aligned} \left( \int_{\mathcal{O}} |\eta H_k(w)|^p y^{\beta-1} dx dy \right)^{1/p} \\ \leq (C\alpha)^\xi \left( \left( 1 + \|\sqrt{y} \nabla \eta\|_{L^\infty(\mathbb{H})}^2 \right)^{1/p} \left( \int_{\text{supp } \eta} |w H'_k(w)|^2 y^{\beta-1} dx dy \right)^{1/2} + |\text{supp } \eta|_{\beta-1}^{1/p} A^\alpha \right). \end{aligned}$$

Because the positive constants  $C$  and  $\xi$  are independent of  $k$ , we may take limit as  $k$  goes to  $\infty$ , in the preceding inequality, and we obtain

$$\begin{aligned} & \left( \int_{\mathcal{O}} |\eta(w^\alpha - A^\alpha)|^p y^{\beta-1} dx dy \right)^{1/p} \\ & \leq (C\alpha)^{\xi+1} \left( \left( 1 + \|\sqrt{y}\nabla\eta\|_{L^\infty(\mathbb{H})}^2 \right)^{1/p} \left( \int_{\text{supp } \eta} |w^\alpha|^2 y^{\beta-1} dx dy \right)^{1/2} + |\text{supp } \eta|_{\beta-1}^{1/p} A^\alpha \right), \end{aligned}$$

which yields

$$\begin{aligned} \left( \int_{\mathcal{O}} |\eta w^\alpha|^p y^{\beta-1} dx dy \right)^{1/p} & \leq (C\alpha)^{\xi+1} \left( \left( 1 + \|\sqrt{y}\nabla\eta\|_{L^\infty(\mathbb{H})}^2 \right)^{1/p} \left( \int_{\text{supp } \eta} |w|^{2\alpha} y^{\beta-1} dx dy \right)^{1/2} \right. \\ & \quad \left. + |\text{supp } \eta|_{\beta-1}^{1/p} A^\alpha \right), \end{aligned}$$

We also have

$$\begin{aligned} A^\alpha & = \left( \frac{1}{|\text{supp } \eta|_{\beta-1}} \int_{\text{supp } \eta} A^{2\alpha} y^{\beta-1} dx dy \right)^{1/2} \\ & \leq \left( \frac{1}{|\text{supp } \eta|_{\beta-1}} \int_{\text{supp } \eta} w^{2\alpha} y^{\beta-1} dx dy \right)^{1/2}. \end{aligned}$$

Combining the last two inequalities gives (4.8). This completes the proof of Claim 4.2.  $\square$

**Step 2** (Moser iteration). The purpose of this step is to apply the Moser iteration technique to  $w$  in (4.7) with a suitable choice of  $\alpha \geq 1$  and of a sequence of non-negative cutoff functions,  $\{\eta_N\}_{N \geq 1} \subset C_0^1(\bar{\mathbb{H}})$ , with support in  $\bar{\mathbb{B}}_{2R}(z_0)$ . We choose  $\{\eta_N\}_{N \in \mathbb{N}}$  as in (3.5) with  $R_N := R(1 + 1/(N+1))$ . Then, (3.6) and (3.7) become

$$\eta_N|_{B_{R_N}(z_0)} \equiv 1, \quad \eta_N|_{B_{R_{N-1}}^c(z_0)} \equiv 0, \quad |\nabla \eta_N| \leq \frac{cN^3}{R^2}, \quad (4.21)$$

where  $c$  is a positive constant independent of  $R$  and  $N$ . For each  $N \geq 0$ , we set  $p_N := 2(p/2)^N$  and  $\alpha_N := (p/2)^N$ . Let  $A_N := \|f\|_{L^s(\text{supp } \eta_N, y^{\beta-1})}$  and  $w_N := u^+ + A_N$  or  $w_N := u^- + A_N$ . Define

$$I(N) := \left( \int_{B_{R_N}(z_0)} |w_N|^{p_N} y^{\beta-1} dx dy \right)^{1/p_N}.$$

Applying the energy estimate (4.8) with  $w = w_N$ ,  $\alpha = \alpha_{N-1}$ , and  $\eta = \eta_N$ , we obtain for all  $N \geq 1$  that

$$I(N) \leq C_0(R, N)I(N-1), \quad (4.22)$$

where we denote

$$C_0(R, N) := [C|\alpha_{N-1}|]^{2(\xi+1)/p_{N-1}} \left( \|\sqrt{y}\nabla\eta_N\|_{L^\infty(\mathbb{H})}^{2/p} + |\text{supp } \eta_N|_{\beta-1}^{1/p-1/2} \right)^{2/p_{N-1}}. \quad (4.23)$$

In the preceding equality,  $C$  is a positive constant depending only on the coefficients of the Heston operator and  $\bar{R}$ . By applying (4.1) and (2.6), there is a constant  $c > 0$  such that

$$c^{-1}R^{4/(p-2)} \leq |B_{2R}(z_0)|_{\beta-1} \leq cR^{4/(p-2)}, \quad \forall 0 < R \leq \bar{R}, \quad (4.24)$$

where we used the fact that  $2(n + \beta - 1) = 4/(p - 2)$  by (2.1). Moreover, by Lemma 2.4, there is a positive constants  $c$  such that  $0 \leq y \leq cR^2$  on  $B_R(z_0)$ , for all  $R \geq 0$ . Consequently, we have

$$\|\sqrt{y}\nabla\eta_N\|_{L^\infty(\mathbb{H})}^{2/p} + |\text{supp } \eta_N|_{\beta-1}^{1/p-1/2} \leq cN^{6/p}R^{-2/p},$$

and so,

$$C_0(R, N) \leq [C|\alpha_{N-1}|N^6]^{p(\xi+1)/p_N} R^{-2/p_N}.$$

Therefore,

$$\begin{aligned} \prod_{N \geq 1} C_0(R, N) &\leq \prod_{N \geq 1} [C|\alpha_{N-1}|N^6]^{p(\xi+1)/p_N} R^{-2/p_N} \\ &\leq C_1 R^{-2 \sum_{N=1}^{\infty} 1/p_N} = C_1 R^{-2/(p-2)} \\ &\leq C_1 |B_{2R}(z_0)|_{\beta-1}^{-1/2}, \quad (\text{by (4.24)}), \end{aligned}$$

where  $C_1$  is a positive constant depending only on the coefficients of the Heston operator,  $\bar{R}$  and  $s$ . By iterating (4.22), we obtain

$$I(+\infty) \leq I(0) \prod_{N \geq 1} C_0(R, N),$$

which gives us (see Lemma A.6)

$$\operatorname{ess\,sup}_{B_R(z_0)} w = I(+\infty) \leq C_1 \left( \frac{1}{|B_{2R}(z_0)|_{\beta-1}} \int_{B_{2R}(z_0)} |w|^2 y^{\beta-1} dx dy \right)^{1/2}. \quad (4.25)$$

Applying (4.25) to both choices of  $w$  in (4.7) yields

$$\begin{aligned} \operatorname{ess\,sup}_{B_R(z_0)} u^+ &\leq C_1 \left[ \left( \frac{1}{|B_{2R}(z_0)|_{\beta-1}} \int_{B_{2R}(z_0)} |u|^2 y^{\beta-1} dx dy \right)^{1/2} + \|f\|_{L^s(B_{2R}(z_0), y^{\beta-1})} \right], \\ \operatorname{ess\,sup}_{B_R(z_0)} u^- &\leq C_1 \left[ \left( \frac{1}{|B_{2R}(z_0)|_{\beta-1}} \int_{B_{2R}(z_0)} |u|^2 y^{\beta-1} dx, dy \right)^{1/2} + \|f\|_{L^s(B_{2R}(z_0), y^{\beta-1})} \right]. \end{aligned}$$

Adding the two estimates gives us the supremum estimate (1.21) .

□

## 5. HÖLDER CONTINUITY FOR SOLUTIONS TO THE VARIATIONAL EQUATION

In this section, we prove Theorem 1.11, that is, local Hölder continuity on a neighborhood of  $\bar{\Gamma}_0$  for solutions  $u$  to the variational equation (1.18). We consider separately the case of the interior boundary points  $z_0 \in \Gamma_0$  and of the “corner points”  $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$ . (While  $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$  is a set of geometric corner points for the domain  $\mathcal{O}$ , the lesson of [16] is that the solution,  $u$ , along  $\Gamma_0$  behaves, in many respects, just as it does in the interior of  $\mathcal{O}$ .) The proof of the second case, for corner points, is easier than the proof of the first case as it does not require an application of the John-Nirenberg inequality. The essential difference between the proof of Theorem 1.11 and the proof of its classical analogue for weak solutions to non-degenerate elliptic equations [28, Theorems 8.27 & 8.29] consists in a modification of the methods of [28, §8.6, §8.9, & §8.10] when deriving our energy estimates (5.12), where we adapt the application of the John-Nirenberg inequality and Poincaré inequality to our framework of weighted Sobolev spaces. Moreover, because the balls defined by the Koch metric,  $d$ , do not have good scaling properties unless they are centered at a point  $z_0 \in \partial\mathbb{H}$  (see Remark 2.11), the Moser iteration technique applies only to such balls. Therefore, the estimate (1.27) holds only for points  $z_0 \in \partial\mathbb{H}$ , and in order to obtain the full Hölder continuity of solutions (1.28), we need to apply a rescaling argument which is

outlined in the last steps of the arguments below. Therefore, boundary Hölder continuity does not follow in the same way as in [28].

We now proceed to the proof of Theorem 1.11, first in §5.1 for the case of points  $z_0 \in \Gamma_0$  and then in §5.2 for points  $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$ .

**5.1. Local Hölder continuity in the interior of the portion of the boundary where  $A$  is degenerate.** We commence with the

*Proof of Theorem 1.11 for points in  $\Gamma_0$ .* Let  $z_0 \in \Gamma_0$  and let  $R$  be small enough such that

$$\mathbb{B}_{4R}(z_0) = B_{4R}(z_0), \quad (5.1)$$

that is,  $4R \leq \min\{\bar{R}, \text{dist}(z_0, \Gamma_1)\}$ , where  $\text{dist}(\cdot, \cdot)$  is the distance function on  $\bar{\mathbb{H}}$  defined by the Koch metric,  $d$ , and we choose  $\bar{R} \leq \bar{R}_0/2$  small enough so that it is less than or equal to the constant  $\bar{R}$  appearing in Lemma 4.1, and such that for all  $z_i = (x_i, y_i) \in B_{\bar{R}}(z_0)$ ,  $i = 1, 2$ , we have

$$0 < y_1 < 1, \quad 0 < y_2 < 1, \quad 0 \leq |z_1 - z_2| < 1, \quad \text{and} \quad 0 \leq d(z_1, z_2) < 1. \quad (5.2)$$

Choose

$$q \in (n + \beta, s), \quad (5.3)$$

$$\omega \in (0, 2), \quad (5.4)$$

and define  $k(R) > 0$  by

$$k \equiv k(R) := \|f\|_{L^q(B_{4R}(z_0), y^{\beta-1})} + (|m_{\bar{R}}| + |M_{\bar{R}}|) R^\omega. \quad (5.5)$$

The remaining steps in the proof will apply to either of the following choices of functions  $w$  defined on  $B_{4R}(z_0)$ ,

$$w = u - m_{4R} + k(R) \quad \text{or} \quad w = M_{4R} - u + k(R), \quad (5.6)$$

but, for concreteness, we choose

$$w = u - m_{4R} + k(R). \quad (5.7)$$

If  $m_{\bar{R}} = M_{\bar{R}} = 0$  or  $m_{4R} = M_{4R} = 0$ , then automatically  $u = 0$  on  $B_{4R}(z_0)$  and (1.27) and (1.28) hold on  $B_{4R}(z_0)$ . Therefore, without loss of generality, we may assume

$$m_{4R} \neq 0 \quad \text{or} \quad M_{4R} \neq 0, \quad (5.8)$$

and  $m_{\bar{R}} \neq 0$  or  $M_{\bar{R}} \neq 0$ . The last assumption implies that

$$k(R) \neq 0, \quad (5.9)$$

by (5.5). Therefore, we notice that both choices of  $w$  in (5.7) are bounded, positive functions.

**Step 1** (Energy estimate for  $w$ ). Let  $\eta \in C_0^1(\bar{\mathbb{H}})$  be a non-negative cutoff function with  $\text{supp } \eta \subseteq \bar{B}_{4R}(z_0)$ . For any  $\alpha \in \mathbb{R}$ ,  $\alpha \neq -1$ , let

$$v := \eta^2 w^\alpha. \quad (5.10)$$

Then,  $v$  is a valid test function in  $H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  by Lemma A.3. Let

$$H(w) := w^{(\alpha+1)/2}, \quad (5.11)$$

and notice that Theorem 1.7 implies that  $H(w)$  is a positive, bounded function, so the following operations are justified. The goal in this step is to prove



**Claim 5.1** (Energy estimate). *There exists a positive constant  $C$  depending only on the coefficients of the Heston operator,  $n$  and  $\bar{R}$ , and there is a positive constant  $\xi$  depending only on  $n$ ,  $\beta$  and  $q$ , such that*

$$\|\eta H(w)\|_{L^p(\mathbb{H}, y^{\beta-1})} \leq C_0(R, \alpha) \|H(w)\|_{L^2(\text{supp } \eta, y^{\beta-1})}, \quad (5.12)$$

where the constant  $C_0(R, \alpha)$  is defined by

$$C_0(R, \alpha) := [C|1 + \alpha|]^{(\xi+1)/p} \left(1 + \|\sqrt{y} \nabla \eta\|_{L^\infty(\mathbb{H})}^2\right)^{1/p}, \quad (5.13)$$

and the constant  $\xi$  is given by

$$\xi \equiv \xi(p, q) := \frac{p(q^* - 1)}{p - 2q^*}, \quad (5.14)$$

where  $q^*$  is the conjugate exponent for  $q$  in (5.3), that is,  $1/q + 1/q^* = 1$ .

The estimate (5.12) will be used in Moser iteration.

Substituting the choice (5.10) of  $v$  in (1.14) and using  $\nabla v = \alpha \eta^2 w^{\alpha-1} \nabla w + 2\eta \nabla \eta w^\alpha$  gives

$$\begin{aligned} 0 &= a(u, v) - (f, v)_{L^2(\mathcal{O}, \mathfrak{w})} \\ &= \frac{\alpha}{2} \int_{\mathbb{H}} \eta^2 w^{\alpha-1} (w_x^2 + 2\rho \sigma w_x w_y + \sigma^2 w_y^2) y \mathfrak{w} \, dx \, dy \\ &\quad + \int_{\mathbb{H}} \eta w^\alpha (w_x \eta_x + \rho \sigma w_x \eta_y + \rho \sigma w_y \eta_x + \sigma^2 w_y \eta_y) y \mathfrak{w} \, dx \, dy \\ &\quad - \int_{\mathbb{H}} \left[ a_1 w_x + \frac{\gamma}{2} (w_x + \rho \sigma w_y) \text{sign}(x) \right] \eta^2 w^\alpha y \mathfrak{w} \, dx \, dy + \int_{\mathbb{H}} (ru - f) \eta^2 w^\alpha \mathfrak{w} \, dx \, dy. \end{aligned}$$

Using (5.11) to compute  $\nabla H(w) = \frac{\alpha+1}{2} w^{(\alpha-1)/2} \nabla w$ , we can rewrite the preceding equation as

$$\begin{aligned} 0 &= \frac{2\alpha}{|1 + \alpha|^2} \int_{\mathbb{H}} \eta^2 [\partial_x H(w)^2 + 2\rho \sigma \partial_x H(w) \partial_y H(w) + \sigma^2 \partial_y H(w)^2] y \mathfrak{w} \, dx \, dy \\ &\quad + \frac{2}{1 + \alpha} \int_{\mathbb{H}} \eta H(w) [\partial_x H(w) \eta_x + \rho \sigma \partial_x H(w) \eta_y + \rho \sigma \partial_y H(w) \eta_x + \sigma^2 \partial_y H(w) \eta_y] y \mathfrak{w} \, dx \, dy \\ &\quad - \frac{2}{1 + \alpha} \int_{\mathbb{H}} \left[ a_1 \partial_x H(w) + \frac{\gamma}{2} (\partial_x H(w) + \rho \sigma \partial_x H(w)) \text{sign}(x) \right] \eta^2 H(w) y \mathfrak{w} \, dx \, dy \\ &\quad + \int_{\mathbb{H}} (rw + r(m_{4R} - k) - f) \eta^2 w^\alpha \mathfrak{w} \, dx \, dy. \end{aligned}$$

Using the uniform ellipticity property (1.16), Hölder's inequality, the fact that  $w \geq k$  by (5.7), and the preceding identity, we see that there is a positive constant  $C$ , depending only on the coefficients of the Heston operator, such that

$$\begin{aligned} \int_{\mathbb{H}} \eta^2 |\nabla H(w)|^2 y \mathfrak{w} \, dx \, dy &\leq C|1 + \alpha| \left[ \int_{\mathbb{H}} (\eta^2 + y |\nabla \eta|^2) w^{\alpha+1} \mathfrak{w} \, dx \, dy \right. \\ &\quad \left. + \int_{\mathbb{H}} \eta^2 |f + r(k - m_{4R})| w^\alpha \mathfrak{w} \, dx \, dy \right], \end{aligned}$$

and hence

$$\begin{aligned} \int_{\mathbb{H}} \eta^2 |\nabla H(w)|^2 y \mathfrak{w} \, dx \, dy &\leq C|1 + \alpha| \left[ \int_{\mathbb{H}} (\eta^2 + y |\nabla \eta|^2) w^{\alpha+1} \mathfrak{w} \, dx \, dy \right. \\ &\quad \left. + \int_{\mathbb{H}} \eta^2 \frac{|f + r(k - m_{4R})|}{k} w^{\alpha+1} \mathfrak{w} \, dx \, dy \right]. \end{aligned}$$

Using definition (1.12) of the weight  $\mathfrak{w}$ , we can rewrite the preceding inequality as

$$\begin{aligned} \int_{\mathbb{H}} \eta^2 |\nabla H(w)|^2 y^\beta dx dy &\leq C|1 + \alpha| \left[ \int_{\mathbb{H}} (\eta^2 + y |\nabla \eta|^2) w^{\alpha+1} y^{\beta-1} dx dy \right. \\ &\quad \left. + \int_{\mathbb{H}} \eta^2 \frac{|f + r(k - m_{4R})|}{k} w^{\alpha+1} y^{\beta-1} dx dy \right], \end{aligned} \quad (5.15)$$

where the constant  $C$  now depends in addition on  $\bar{R}$ . By Hölder's inequality, we have

$$\begin{aligned} \int_{\mathbb{H}} \eta^2 \frac{|f + r(k - m_{4R})|}{k} w^{\alpha+1} y^{\beta-1} dx dy &\leq \left( \int_{\text{supp } \eta} \left| \frac{f + r(k - m_{4R})}{k} \right|^q y^{\beta-1} dx dy \right)^{1/q} \\ &\quad \times \left( \int_{\mathbb{H}} |\eta w^{(\alpha+1)/2}|^{2q^*} y^{\beta-1} dx dy \right)^{1/q^*}. \end{aligned} \quad (5.16)$$

From our definition of  $k$  in (5.5), there is a positive constant  $C$ , depending only on  $\bar{R}$  and  $\beta$ , such that

$$\left( \int_{\text{supp } \eta} \left| \frac{f + r(k - m_{4R})}{k} \right|^q y^{\beta-1} dx dy \right)^{1/q} \leq C. \quad (5.17)$$

From inequalities (5.15), (5.16) and (5.17), we obtain

$$\begin{aligned} \int_{\mathbb{H}} \eta^2 |\nabla H(w)|^2 y^\beta dx dy &\leq C|1 + \alpha| \left[ \int_{\mathbb{H}} (\eta^2 + y |\nabla \eta|^2) w^{\alpha+1} y^{\beta-1} dx dy \right. \\ &\quad \left. + \left( \int_{\mathbb{H}} |\eta w^{(\alpha+1)/2}|^{2q^*} y^{\beta-1} dx dy \right)^{1/q^*} \right], \end{aligned} \quad (5.18)$$

where the positive constant  $C$  depends on the coefficients of the Heston operator and  $\bar{R}$ . We apply Lemma 2.2 to  $\eta H(w)$  and we have

$$\begin{aligned} \int_{\mathbb{H}} |\eta H(w)|^p y^{\beta-1} dx dy &\leq \left( \int_{\mathbb{H}} \eta^2 |H(w)|^2 y^{\beta-1} dx dy \right)^{(p-2)/2} \int_{\mathbb{H}} |\nabla(\eta H(w))|^2 y^\beta dx dy \\ &\leq \left( \int_{\mathbb{H}} \eta^2 |H(w)|^2 y^{\beta-1} dx dy \right)^{(p-2)/2} \\ &\quad \times \left( \int_{\mathbb{H}} |\nabla \eta|^2 |H(w)|^2 y^\beta dx dy + \int_{\mathbb{H}} \eta^2 |\nabla H(w)|^2 y^\beta dx dy \right). \end{aligned}$$

Combining the preceding inequality with (5.18), we obtain

$$\begin{aligned} &\int_{\mathbb{H}} |\eta H(w)|^p y^{\beta-1} dx dy \\ &\leq C|1 + \alpha| \left( \int_{\mathbb{H}} \eta^2 |H(w)|^2 y^{\beta-1} dx dy \right)^{(p-2)/2} \\ &\quad \times \left[ \int_{\mathbb{H}} (\eta^2 + y |\nabla \eta|^2) |H(w)|^2 y^{\beta-1} dx dy + \left( \int_{\mathbb{H}} |\eta H(w)|^{2q^*} y^{\beta-1} dx dy \right)^{1/q^*} \right], \end{aligned}$$

and thus

$$\begin{aligned}
& \int_{\mathbb{H}} |\eta H(w)|^p y^{\beta-1} dx dy \\
& \leq C|1 + \alpha| \left( 1 + \|\sqrt{y} \nabla \eta\|_{L^\infty(\mathbb{H})}^2 \right) \left( \int_{\text{supp } \eta} \eta^2 |H(w)|^2 y^{\beta-1} dx dy \right)^{p/2} \\
& \quad + C|1 + \alpha| \left( \int_{\mathbb{H}} \eta^2 |H(w)|^2 y^{\beta-1} dx dy \right)^{(p-2)/2} \left( \int_{\mathbb{H}} |\eta H(w)|^{2q^*} y^{\beta-1} dx dy \right)^{1/q^*}.
\end{aligned} \tag{5.19}$$

From our assumption (5.3) that  $q > n + \beta$ , we have  $2 < 2q^* < p$ . Since  $q < \infty$  implies  $q^* > 1$ , while  $q > n + \beta$  implies

$$q^* < (n + \beta)/(n + \beta - 1), \tag{5.20}$$

and thus  $2q^* < p$  by (2.1). Hence, we may apply the interpolation inequality [28, Inequality (7.10)], for any  $\varepsilon > 0$ , to give

$$\|\eta H(w)\|_{L^{2q^*}(\mathbb{H}, y^{\beta-1})} \leq \varepsilon \|\eta H(w)\|_{L^p(\mathbb{H}, y^{\beta-1})} + \varepsilon^{-\xi} \|\eta H(w)\|_{L^2(\mathbb{H}, y^{\beta-1})},$$

where  $\xi$  is given by (5.14). We need the preceding inequality in the form

$$\begin{aligned}
\left( \int_{\mathbb{H}} |\eta H(w)|^{2q^*} y^{\beta-1} dx dy \right)^{1/q^*} &= \|\eta H(w)\|_{L^{2q^*}(\mathbb{H}, y^{\beta-1})}^2 \\
&\leq 2\varepsilon^2 \|\eta H(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^2 + 2\varepsilon^{-2\xi} \|\eta H(w)\|_{L^2(\mathbb{H}, y^{\beta-1})}^2.
\end{aligned}$$

Applying the preceding inequality in (5.19), we obtain

$$\begin{aligned}
& \|\eta H(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^p \\
& \leq C|1 + \alpha| \left( 1 + \|\sqrt{y} \nabla \eta\|_{L^\infty(\mathbb{H})}^2 \right) \|H(w)\|_{L^2(\text{supp } \eta, y^{\beta-1})}^p \\
& \quad + C|1 + \alpha| \|\eta H(w)\|_{L^2(\text{supp } \eta, y^{\beta-1})}^{p-2} \left( \varepsilon^2 \|\eta H(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^2 + \varepsilon^{-2\xi} \|\eta H(w)\|_{L^2(\mathbb{H}, y^{\beta-1})}^2 \right).
\end{aligned}$$

Recombining terms, we see that

$$\begin{aligned}
& \|\eta H(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^p \\
& \leq C|1 + \alpha| \left( 1 + \varepsilon^{-2\xi} \right) \left( 1 + \|\sqrt{y} \nabla \eta\|_{L^\infty(\mathbb{H})}^2 \right) \|H(w)\|_{L^2(\text{supp } \eta, y^{\beta-1})}^p \\
& \quad + C|1 + \alpha| \varepsilon^2 \|\eta H(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^2 \|\eta H(w)\|_{L^2(\mathbb{H}, y^{\beta-1})}^{p-2}.
\end{aligned}$$

To bound the last term in the preceding inequality, we apply Young's inequality with the conjugate exponents  $(p/2, p/(p-2))$  to give

$$\|\eta H(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^2 \|\eta H(w)\|_{L^2(\mathbb{H}, y^{\beta-1})}^{p-2} \leq \frac{2}{p} \|\eta H(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^p + \frac{p-2}{p} \|\eta H(w)\|_{L^2(\mathbb{H}, y^{\beta-1})}^p.$$

Thus,

$$\begin{aligned}
\|\eta H(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^p &\leq C|1 + \alpha| \left( 1 + (\varepsilon^2 + \varepsilon^{-2\xi}) \right) \left( 1 + \|\sqrt{y} \nabla \eta\|_{L^\infty(\mathbb{H})}^2 \right) \|H(w)\|_{L^2(\text{supp } \eta, y^{\beta-1})}^p \\
&\quad + C|1 + \alpha| \varepsilon^2 \|\eta H(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^p.
\end{aligned}$$

By choosing  $\varepsilon = 1/(2C|1 + \alpha|)^{1/2}$  and taking roots of order  $p$ , we find that

$$\|\eta H(w)\|_{L^p(\mathbb{H}, y^{\beta-1})} \leq [C|1 + \alpha|]^{(\xi+1)/p} \left( 1 + \|\sqrt{y} \nabla \eta\|_{L^\infty(\mathbb{H})}^2 \right)^{1/p} \|H(w)\|_{L^2(\text{supp } \eta, y^{\beta-1})},$$

which is equivalent to (5.12) and (5.13).

**Step 2** (Moser iteration with negative power). In this step we apply the Moser iteration technique starting with a suitable  $\alpha = \alpha_0 < -1$  in (5.12) with functions  $w$  in (4.7). Let  $\{\eta_N\}_{N \in \mathbb{N}}$  be the sequence of cutoff functions considered in Step 2 in the proof of Theorem 1.7. Let  $\alpha_0 < -1$ ,  $p_0 := \alpha_0 + 1$ ,  $p_N := p_0(p/2)^N$ , where  $p$  is as in (2.1), and  $\alpha_N + 1 := p_N$ . We notice that  $p_N \rightarrow -\infty$  as  $N$  increases. Set

$$I(N) := \left( \int_{B_{R_N}(z_0)} |w|^{p_N} y^{\beta-1} dx dy \right)^{1/p_N}.$$

By applying (5.12) with  $w = u - m_{4R} + k$ ,  $\alpha = \alpha_{N-1}$ , and  $\eta = \eta_N$ , we obtain for all  $N \geq 1$ ,

$$\left( \int_{\mathbb{H}} \eta_N^p w^{(\alpha_{N-1}+1)p/2} y^{\beta-1} dx dy \right)^{1/p} \leq C_0(R, \alpha) \left( \int_{\mathbb{H}} |\eta_N w^{(\alpha_{N-1}+1)/2}|^2 y^{\beta-1} dx dy \right)^{1/2}.$$

Since  $(\alpha_{N-1} + 1)p/2 = p_N$  and  $\alpha_{N-1} + 1 = p_{N-1}$ , we can write the preceding inequality as

$$\left( \int_{B_{R_N}(z_0)} |w|^{p_N} y^{\beta-1} dx dy \right)^{1/p} \leq C_0(R, \alpha) \left( \int_{B_{R_{N-1}}(z_0)} |w|^{p_{N-1}} y^{\beta-1} dx dy \right)^{1/2}.$$

Taking roots of order  $p/p_N$  and noticing that  $p/p_N < 0$ , we obtain

$$I(N) \geq C_1(R, N) I(N-1), \quad (5.21)$$

where  $C_1(R, N)$  is given by

$$C_1(R, N) := [C|p_{N-1}|]^{(\xi+1)/p_N} \left( 1 + \|\sqrt{y} \nabla \eta_m\|_{L^\infty(\mathbb{H})}^2 \right)^{1/p_N}.$$

and  $C$  is a positive constant, independent of  $R$  and  $N$ , depending only on the coefficients of the Heston operator and  $\bar{R}$ . From Lemma 2.4, the bound on  $|\nabla \eta_N|$  and the fact that  $0 < R < 1$ , we obtain

$$1 + \|\sqrt{y} \nabla \eta_m\|_{L^\infty(\mathbb{H})}^2 \leq cN^6 R^{-2},$$

for some positive constant  $c$ , and so, we may assume without loss of generality

$$C_1(R, N) = [C|p_{N-1}|N^6]^{(\xi+1)/p_N} R^{-2/p_N}. \quad (5.22)$$

We notice that

$$\prod_{N \geq 1} C_1(R, N) = C_2 R^{4/(|p_0|(p-2))} < \infty,$$

where  $C_2$  depends at most on the coefficients of the Heston operator,  $\bar{R}$  and  $q$ . From (4.24), we know that for some constant  $c > 0$  we have  $|B_{2R}(z_0)|_{\beta-1} \geq cR^{4/(p-2)}$ . Thus,

$$\prod_{N \geq 1} C_1(R, N) \geq C_2 |B_{2R}(z_0)|_{\beta-1}^{1/|p_0|}.$$

By iterating (5.21), we obtain  $I(-\infty) \geq I(0) \prod_{N \geq 1} C_0(R, N)$ , which gives us

$$\operatorname{ess\,inf}_{B_R(z_0)} w = I(-\infty) \geq C_2 \left( \frac{1}{|B_{2R}(z_0)|_{\beta-1}} \int_{B_{2R}(z_0)} |w|^{p_0} y^{\beta-1} dx dy \right)^{1/p_0}. \quad (5.23)$$

**Step 3** (Application of Theorem 3.1). The purpose of this step is to show that we may apply Theorem 3.1 to  $w$  with  $S_r = B_{(2+r)R}(z_0)$ ,  $0 \leq r \leq 1$ , and  $\theta_0 = \theta_1 = 1$ . By Proposition 3.2, we find that  $w$  satisfies the inequalities (3.2), so it remains to show that (3.3) holds for  $\log w$ . For  $A$  as defined in (3.3) and  $S_r = \mathbb{B}_{(2+r)R}(z_0) = B_{(2+r)R}(z_0)$ , writing  $B_{(2+r)R}(z_0)$  in place of  $B_{(2+r)R}(z_0)$  for brevity, we have by Hölder's inequality that

$$A \leq \sup_{0 \leq r \leq 1} \inf_{c \in \mathbb{R}} \left( \frac{1}{|B_{(2+r)R}(z_0)|^{\beta-1}} \int_{B_{(2+r)R}(z_0)} |\log w - c|^2 y^{\beta-1} dx dy \right)^{1/2},$$

and so, Corollary 2.6 gives us

$$A \leq \sup_{0 \leq r \leq 1} ((2+r)R)^2 \left( \frac{1}{|B_{(2+r)R}(z_0)|^\beta} \int_{B_{(2+r)R}(z_0)} |\nabla \log w|^2 y^\beta dx dy \right)^{1/2}. \quad (5.24)$$

Let  $\eta \in C_0^1(\bar{\mathbb{H}})$  be a non-negative cutoff function such that  $\eta = 1$  on  $B_{(2+r)R}(z_0)$ ,  $\eta = 0$  outside  $B_{4R}(z_0)$ , and  $|\nabla \eta| \leq C/R^2$ . We choose  $v = \eta^2/w$ , where  $w$  is given by (5.6), or (5.7) for concreteness, and notice that  $v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  by Lemma A.3. With this choice of  $v$  as a test function in the variational equation (1.14) satisfied by  $u$ , we obtain

$$\begin{aligned} (f, v)_{L^2(\mathcal{O}, \mathfrak{w})} &= a(u, v) \\ &= -\frac{1}{2} \int_{\mathcal{O}} \frac{\eta^2}{w^2} (w_x^2 + 2\rho\sigma w_x w_y + \sigma^2 w_y^2) y \mathfrak{w} dx dy \\ &\quad + \int_{\mathcal{O}} \frac{\eta}{w} [w_x \eta_x + \rho\sigma(w_x \eta_y + w_y \eta_x) + \sigma^2 w_y \eta_y] y \mathfrak{w} dx dy \\ &\quad - \int_{\mathcal{O}} \frac{\eta^2}{w} \left[ \frac{\gamma \operatorname{sign}(x)}{2} (w_x + \rho\sigma w_y) + a_1 w_x \right] y \mathfrak{w} dx dy \\ &\quad + \int_{\mathcal{O}} r \eta^2 \frac{u}{w} \mathfrak{w} dx dy. \end{aligned}$$

Using the uniform ellipticity property and Hölder's inequality, we obtain there is a positive constant  $C$ , depending only on the coefficients of the Heston operator and  $\bar{R}$ , such that

$$\int_{\mathcal{O}} \eta^2 |\nabla \log w|^2 y^\beta dx dy \leq C \int_{\mathcal{O}} (|\nabla \eta|^2 + \eta^2) y^\beta dx dy + C \int_{\mathcal{O}} \eta^2 \frac{|f| + |u|}{w} y^{\beta-1} dx dy. \quad (5.25)$$

From Lemma 2.4, assumption (5.1) and the fact that  $|\nabla \eta| \leq C/R^2$ , we have

$$\begin{aligned} \int_{\mathcal{O}} (|\nabla \eta|^2 + \eta^2) y^\beta dx dy &\leq C R^{-4} R^{2(n+\beta)} \\ &\leq C ((2+r)R)^{-4} |B_{(2+r)R}(z_0)|^\beta. \end{aligned} \quad (5.26)$$

Using the definition (5.5) of  $k(R)$  and Hölder's inequality, we obtain

$$\begin{aligned} \int_{\mathcal{O}} \eta^2 \frac{|f| + |u|}{w} y^{\beta-1} dx dy &\leq \frac{1}{\|f\|_{L^q(B_{4R}(z_0), y^{\beta-1})}} \|f\|_{L^q(B_{4R}(z_0), y^{\beta-1})} R^{2(n+\beta-1)/q^*} \\ &\quad + \frac{1}{R^\omega} R^{2(n+\beta-1)}, \end{aligned}$$

and thus

$$\int_{\mathcal{O}} \eta^2 \frac{|f| + |u|}{w} y^{\beta-1} dx dy \leq C \left( R^{2(n+\beta-1)/q^*} + R^{2(n+\beta-1)-\omega} \right). \quad (5.27)$$

The condition  $q > n + \beta$  implies

$$2(n + \beta - 1)/q^* - 2(n + \beta) > -4, \quad (5.28)$$

since  $1/q + 1/q^* = 1$ . Also, because  $\omega$  is chosen in  $(0, 2)$ , we obviously have

$$-2 - \omega > -4. \quad (5.29)$$

Using (5.28) and (5.29), and  $0 < R \leq \bar{R}$ , we obtain in inequality (5.27) that there is a positive constant  $C$ , depending only on the coefficients of the Heston operator and  $\bar{R}$ , such that

$$\begin{aligned} \int_{\mathcal{O}} \eta^2 \frac{|f| + |u|}{w} y^{\beta-1} dx dy &\leq C ((2+r)R)^{2(n+\beta)-4} \\ &\leq C ((2+r)R)^{-4} |B_{(2+r)R}(z_0)|_{\beta}. \end{aligned} \quad (5.30)$$

In the last inequality, we used Lemma 2.4 and (5.1). By combining equations (5.25), (5.26) and (5.30), we obtain

$$\int_{B_{(2+r)R}(z_0)} |\nabla \log w|^2 y^{\beta} dx dy \leq C ((2+r)R)^{-4} |B_{(2+r)R}(z_0)|_{\beta}.$$

Then, it immediately follows that the right hand side of (5.24) is finite, and so, (3.3) holds for  $\log w$ .

**Step 4** (Proof of inequality (1.27)). In the previous step we showed that Theorem 3.1 applies to  $w$  with  $\theta_0 = \theta_1 = 1$ . Hence, there is a constant  $C > 0$ , depending only on the coefficients of the Heston operator and  $\bar{R}$ , but independent of  $R$  and  $w$ , such that

$$\left( \frac{1}{|B_{2R}(z_0)|_{\beta-1}} \int_{B_{2R}(z_0)} |w| y^{\beta-1} dx dy \right) \leq C \left( \frac{1}{|B_{2R}(z_0)|_{\beta-1}} \int_{B_{2R}(z_0)} |w|^{-1} y^{\beta-1} dx dy \right)^{-1}. \quad (5.31)$$

From (5.23) and Lemma A.6, we obtain

$$\operatorname{ess\,inf}_{B_R(z_0)} w = I(-\infty) \geq C \left( \frac{1}{|B_{2R}(z_0)|_{\beta-1}} \int_{B_{2R}(z_0)} |w| y^{\beta-1} dx dy \right). \quad (5.32)$$

We now choose  $w = u - m_{4R} + k$  and  $w = M_{4R} - u + k$  in (5.32). By adding the following two inequalities

$$\begin{aligned} m_R - m_{4R} + k(R) &= \operatorname{ess\,inf}_{B_R(z_0)} (u - m_{4R} + k(R)) \\ &\geq \frac{C}{|B_{2R}(z_0)|_{\beta-1}} \int_{B_{2R}(z_0)} (u - m_{4R} + k(R)) y^{\beta-1} dx dy \\ &\geq \frac{C}{|B_{2R}(z_0)|_{\beta-1}} \int_{B_{2R}(z_0)} (u - m_{4R}) y^{\beta-1} dx dy, \\ M_{4R} - M_R + k(R) &= \operatorname{ess\,inf}_{B_{2R}} (M_{4R} - u + k(R)) \\ &\geq \frac{C}{|B_{2R}(z_0)|_{\beta-1}} \int_{B_{2R}(z_0)} (M_{4R} - u + k(R)) y^{\beta-1} dx dy \\ &\geq \frac{C}{|B_{2R}(z_0)|_{\beta-1}} \int_{B_{2R}(z_0)} (M_{4R} - u) y^{\beta-1} dx dy, \end{aligned}$$

we obtain

$$(M_{4R} - m_{4R}) - (M_R - m_R) + 2k(R) \geq C (M_{4R} - m_{4R}).$$

Without loss of generality, we may assume  $C < 1$  (if not, we can make  $C$  smaller on the right-hand side of the preceding inequality). Therefore, the preceding inequality can be rewritten in the form

$$\operatorname{osc}_{B_R(z_0)} u \leq C \operatorname{osc}_{B_{4R}(z_0)} u + 2k(R). \quad (5.33)$$

Because  $q \in (n + \beta, s)$  by (5.3) and  $f \in L^s(B_{\bar{R}}(z_0), \mathfrak{w})$  for some  $s > n + \beta$ , by hypothesis in Theorem 1.11, Hölder's inequality yields

$$\|f\|_{L^q(B_{4R}(z_0), y^{\beta-1})} \leq CR^{2(n+\beta-1)\frac{s-q}{sq}} \|f\|_{L^s(B_{\bar{R}}(z_0), y^{\beta-1})}.$$

Let

$$\nu := \min \left\{ \omega, 2(n + \beta - 1) \frac{s - q}{sq} \right\}.$$

Consequently, from (5.5), we see that there is a positive constant  $C$ , depending only on  $n = 2$  and  $\beta$ , such that

$$k(R) \leq C \left( \|f\|_{L^s(B_{\bar{R}}(z_0), y^{\beta-1})} + |m_{\bar{R}}| + |M_{\bar{R}}| \right) R^\nu. \quad (5.34)$$

Therefore, by applying [28, Lemma 8.23] to (5.33) and using the inequality (5.34), we find that there is a positive constant  $C$  depending on the coefficients of the Heston operator, the constant  $\bar{R}$ ,  $\|f\|_{L^s(B_{\bar{R}}(z_0), y^{\beta-1})}$ , and  $\|u\|_{L^\infty(B_{\bar{R}}(z_0))}$ , and there is a constant  $\alpha_0 \in (0, 1)$ , depending on  $s, n$  and  $\beta$ , such that

$$\operatorname{osc}_{B_R(z_0)} u \leq CR^{\alpha_0},$$

which is the desired inequality (1.27). Because  $\bar{R}$  depends on the uniform interior and exterior cone condition with cone  $K$ , and is chosen such that  $\bar{R} \leq \bar{R}_0/2$ , we notice that the constant  $C$  depends on the coefficients of the Heston operator,  $A$ , and on  $\bar{R}_0$ ,  $K$ ,  $\|f\|_{L^s(B_{\bar{R}_0}(z_0), y^{\beta-1})}$ , and  $\|u\|_{L^\infty(B_{\bar{R}_0}(z_0))}$ .

**Step 5** (Proof of inequality (1.28)). We prove the estimate (1.28) for points  $z_1, z_2 \in \bar{B}_R(z_0)$ , where  $R$  satisfies

$$0 < 8R \leq \min\{\bar{R}, \operatorname{dist}(z_0, \Gamma_1)\}, \quad (5.35)$$

where  $\operatorname{dist}(\cdot, \cdot)$  is the distance function defined by the Koch metric. Condition (5.35) implies that for any  $z \in B_R(z_0)$ , we have that (5.1) holds for  $B_R(z)$ , and so estimate (1.27) applies on such balls. In particular, for any points  $(x_1, y_1), (x_1, 0), (x_2, 0) \in \bar{B}_R(z_0)$ , the estimate (1.27) gives

$$\begin{aligned} |u(x_1, y_1) - u(x_1, 0)| &\leq Cd((x_1, y_1), (x_1, 0))^{\alpha_0}, \\ |u(x_1, 0) - u(x_2, 0)| &\leq Cd((x_1, 0), (x_2, 0))^{\alpha_0}. \end{aligned} \quad (5.36)$$

Notice that we have the simple identities

$$\begin{aligned} d((x_1, y_1), (x_1, 0)) &= \sqrt{y_1/2}, \\ d((x_1, 0), (x_2, 0)) &= \sqrt{|x_1 - x_2|}, \end{aligned} \quad (5.37)$$

and so, we can rewrite (5.36) in the form

$$\begin{aligned} |u(x_1, y_1) - u(x_1, 0)| &\leq C|y_1|^{\alpha_0/2}, \\ |u(x_1, 0) - u(x_2, 0)| &\leq C|x_1 - x_2|^{\alpha_0/2}. \end{aligned} \quad (5.38)$$

The idea of inequality (1.28) the proof now follows [16, Corollary I.9.7 & Theorem I.9.8], but with certain differences which we outline for clarity. Let  $\varepsilon \in (0, 1/8)$  be fixed and consider the following two cases.



**Case 1** (Pairs of points in  $B_R(z_0)$  obeying (5.39)). Let  $z_i = (x_i, y_i) \in B_R(z_0)$ ,  $i = 1, 2$ , be such that

$$|z_1 - z_2| \geq \varepsilon(y_1^2 + y_2^2). \quad (5.39)$$

From (5.2), we can find a positive constant  $C$  such that

$$|x_1 - x_2| \leq Cd(z_1, z_2). \quad (5.40)$$

Using our current assumption (5.39), in addition to (5.2), we also have

$$d(z_1, z_2) \geq \varepsilon C y_i^2, \quad i = 1, 2,$$

and so, there exists a positive constant  $C$ , depending on  $\varepsilon$ , such that

$$y_i \leq Cd(z_1, z_2)^{1/2}, \quad i = 1, 2. \quad (5.41)$$

Denote  $z'_i = (x_i, 0)$ , for  $i = 1, 2$ . Applying (5.40) and (5.41) in (5.38), we obtain

$$\begin{aligned} |u(z_i) - u(z'_i)| &\leq Cd(z_1, z_2)^{\alpha_0/4}, \quad i = 1, 2, \\ |u(z'_1) - u(z'_2)| &\leq Cd(z_1, z_2)^{\alpha_0/2}, \end{aligned}$$

and hence, using (5.2),

$$\begin{aligned} |u(z_1) - u(z_2)| &\leq |u(z_1) - u(z'_1)| + |u(z'_1) - u(z'_2)| + |u(z_2) - u(z'_2)| \\ &\leq Cd(z_1, z_2)^{\alpha_0/4}, \end{aligned}$$

that is,

$$|u(z_1) - u(z_2)| \leq Cd(z_1, z_2)^{\alpha_0/4}. \quad (5.42)$$

This concludes the proof of Case 1. Therefore, the estimate (1.28) holds in the special case  $|z_1 - z_2| \geq \varepsilon(y_1^2 + y_2^2)$ .

**Case 2** (Pairs of points in  $B_R(z_0)$  obeying (5.43)). Now we consider points  $z_i = (x_i, y_i) \in B_R(z_0)$ ,  $i = 1, 2$ , such that

$$|z_1 - z_2| < \varepsilon(y_1^2 + y_2^2). \quad (5.43)$$

By scaling and using interior Hölder estimates [28, Theorem 8.22], we show that the estimate (1.28) also holds in this case. We proceed by analogy with the proofs of [16, Theorems I.9.1–4 & Corollary I.9.7]. We may assume without loss of generality that

$$1 > y_2 \geq y_1 \quad \text{and} \quad x_2 = 0. \quad (5.44)$$

Let  $a = y_2$ . We consider the function  $v$  defined by rescaling,

$$u(x, y) =: v(x/a, y/a).$$

The rescaling  $z \mapsto z' = z/a$  maps  $\mathbb{E}_{y_2/2}(z_2)$  into  $\mathbb{E}_{1/2}(z'_2)$ . Recall that  $\mathbb{E}_r(z)$  denotes the Euclidean ball centered at  $z$  of radius  $r$  relative to  $\mathbb{H}$  (Definition 2.3). From our assumptions (5.2), (5.43) and the choice of  $\varepsilon \in (0, 1/8)$ , we see that

$$|z'_1 - z'_2| \leq 2\varepsilon y_2 < 1/4, \quad (5.45)$$

and so  $z'_1 \in \mathbb{E}_{1/4}(z'_2)$ . From [15, Theorem 5.10], we know that  $u \in H_{\text{loc}}^2(B_{\bar{R}}(z_0))$ , and so by direct calculation, we conclude that  $v(z')$  solves

$$\tilde{A}v(z') = af(az') \quad \text{on } \mathbb{E}_{1/2}(z'_2),$$

where we define

$$(\tilde{A}v)(z') := \frac{1}{2}y'(v_{xx} + 2\rho\sigma v_{xy} + \sigma^2 v_{yy})(z') + (r - q - ay'/2)v_x(z') \\ + \kappa(\theta - ay')v_y(z') - arv(z').$$

On the ball  $\mathbb{E}_{1/2}(z'_2)$ , the operator  $\tilde{A}$  is uniformly elliptic with bounded coefficients. Moreover, there is a positive constant  $M$ , depending only on the coefficients of the Heston operator, such that for all  $a \in (0, 1)$ ,  $M$  is a uniform bound on the  $L^\infty(\mathbb{E}_{1/2}(z'_2))$ -norm of the coefficients of  $\tilde{A}$ . For brevity, we denote  $f_a(z') := af(az')$ . By [28, Theorem 8.22], there are positive constants  $C$  and  $\alpha_0 \in (0, 1)$ , depending only on the  $L^\infty(\mathbb{E}_{1/2}(z'_2))$ -bounds of the coefficients, such that

$$\operatorname{osc}_{\mathbb{E}_R(z'_2)} v \leq CR^{\alpha_0} \left( \|v\|_{L^\infty(\mathbb{E}_{1/2}(z'_2))} + \|f_a\|_{L^s(\mathbb{E}_{1/2}(z'_2))} \right), \quad \forall R \in (0, 1/2], \quad (5.46)$$

because  $s$  was assumed to satisfy  $s > 2n$  (recall that  $n = 2$ ). We see that

$$\|v\|_{L^\infty(\mathbb{E}_{1/2}(z'_2))} = \|u\|_{L^\infty(\mathbb{E}_{y_2/2}(z_2))} \leq \|u\|_{L^\infty(B_{\bar{R}}(z_0))}, \quad (5.47)$$

where we used the fact that  $\mathbb{E}_{y_2/2}(z_2) \subseteq B_{\bar{R}}(z_0)$ , which in turn follows from the requirement  $4R \leq \bar{R}$  in the hypotheses of Theorem 1.11. We also have

$$\|f_a\|_{L^s(\mathbb{E}_{1/2}(z'_2))}^s = \int_{\mathbb{E}_{1/2}(z'_2)} |af(az')|^s dx' dy' = \int_{\mathbb{E}_{y_2/2}(z_2)} |f(z)|^s a^{s-n} dx dy,$$

that is,

$$\|f_a\|_{L^s(\mathbb{E}_{1/2}(z'_2))}^s = \int_{\mathbb{E}_{y_2/2}(z_2)} |f(z)|^s a^{s-n} dx dy. \quad (5.48)$$

Using the fact that  $y_2/2 \leq y \leq 3y_2/2$  for all  $z = (x, y) \in \mathbb{E}_{y_2/2}(z_2)$ , assumption (5.2), and the fact that  $s > n + \beta$  by hypothesis of Theorem 1.11, the estimate (5.48) yields

$$\|f_a\|_{L^s(\mathbb{E}_{1/2}(z'_2))}^s \leq C \int_{B_{\bar{R}}(z_0)} |f(z)|^s y^{\beta-1} dx dy, \quad (5.49)$$

where  $C$  is a positive constant depending only on  $\beta$ . Applying (5.47) and (5.49) in (5.46) yields

$$\operatorname{osc}_{\mathbb{E}_R(z'_2)} v \leq CR^{\alpha_0} \left( \|u\|_{L^\infty(B_{\bar{R}}(z_0))} + \|f\|_{L^s(B_{\bar{R}}(z_0))} \right), \quad \forall R \in (0, 1/2].$$

In particular, because  $z'_1 \in \mathbb{E}_{1/2}(z'_2)$ , we see that

$$|v(z'_1) - v(z'_2)| \leq C|z'_1 - z'_2|^{\alpha_0},$$

where the positive constant  $C$  now depends on  $\|u\|_{L^\infty(B_{4\bar{R}}(z_0))}$  and  $\|f\|_{L^s(B_{4\bar{R}}(z_0), w)}$ . By rescaling back, we obtain

$$|u(z_1) - u(z_2)| \leq C \left( \frac{|z_1 - z_2|}{y_2} \right)^{\alpha_0}. \quad (5.50)$$

Using the following sequence of inequalities,

$$\begin{aligned} \frac{|z_1 - z_2|^2/y_2^2}{d(z_1, z_2)} &\leq \frac{|z_1 - z_2|^2 \sqrt{y_1 + y_2 + |z_1 - z_2|}}{y_2^2 |z_1 - z_2|} \\ &= \frac{|z_1 - z_2|}{y_2^2} \sqrt{y_1 + y_2 + |z_1 - z_2|} \\ &\leq 2\varepsilon \sqrt{y_1 + y_2 + |z_1 - z_2|} \\ &\leq 1 \quad (\text{by (5.2) and } \varepsilon \in (0, 1/8)), \end{aligned}$$

we therefore have

$$\frac{|z_1 - z_2|}{y_2} \leq d(z_1, z_2)^{1/2}. \quad (5.51)$$

Consequently, (5.50) gives us

$$|u(z_1) - u(z_2)| \leq Cd(z_1, z_2)^{\alpha_0/2}.$$

This concludes the proof of Case 2.

By combining Cases 1 and 2, we find that, for any  $z_1, z_2 \in B_R(z_0)$  and  $R$  satisfying (5.1), (5.2) and (5.35) (see Lemma 4.1 regarding the expressions for the upper bound for  $R$  in the hypotheses of Theorem 1.11), we have

$$|u(z_1) - u(z_2)| \leq C_2 d(z_1, z_2)^{\alpha_2}, \quad (5.52)$$

where  $C_2$  and  $\alpha_2$  are constants with the dependencies stated in Theorem 1.11 for  $C_1$  and  $\alpha_1$ , respectively. Notice that this inequality is not as strong as the inequality (1.28), which holds for all  $z_1, z_2 \in B_{\bar{R}}(z_0)$ . We shall explain in §5.2 how the inequality (1.28) is obtained after the solutions  $u$  are shown to be Hölder continuous at points  $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$ , which we also prove in §5.2.  $\square$

## 5.2. Hölder continuity on neighborhoods of the corner points of the portion of the boundary where $A$ is degenerate. We now have

*Proof of Theorem 1.11 for points in  $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ .* Suppose  $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$ . We let  $\bar{R}$  be as in the proof of Theorem 1.11 for points in  $\Gamma_0$ , in §5.1. From the standard theory of non-degenerate elliptic partial differential equations (for example, [28, Theorem 8.30]), we know that

$$u \in C(\bar{B}_{\bar{R}}(z_0) \cap \mathbb{H}) \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial B_{\bar{R}}(z_0) \cap \Gamma_1. \quad (5.53)$$

Recalling that  $u^+ = \max\{u, 0\}$  and  $u^- = \max\{-u, 0\}$  denote the positive and negative parts of  $u$ , respectively, we have that  $u^\pm \in C(\bar{B}_{\bar{R}}(z_0) \cap \mathbb{H})$  and  $u^\pm = 0$  along the portion of the boundary  $\partial B_{\bar{R}}(z_0) \cap \Gamma_1$ .

Our goal is first to prove that there are constants  $C$ , depending only on the coefficients of the Heston operator, the cone  $K$ ,  $n$ ,  $s$ ,  $\bar{R}_0$ ,  $\|f\|_{L^s(B_{\bar{R}_0}(z_0), y^{\beta-1})}$  and  $\|u\|_{L^\infty(B_{\bar{R}_0}(z_0))}$ , and  $\alpha_0$ , depending only on  $n$ ,  $s$  and  $\beta$ , such that

$$\operatorname{osc}_{B_R(z_0)} u^\pm \leq CR^{\alpha_0}, \quad \forall 0 < 8R \leq \bar{R}, \quad (5.54)$$

which obviously implies that (1.27) holds for  $u$ , for possibly a different constant  $C$  with the same dependency as above.

Our proof uses the same method as in the case of points in  $\Gamma_0$  but a choice of  $w$  which is different from that of (4.7), and a choice of test function  $v$  which is different from that of (5.10). Moreover, we do not need to appeal to the John-Nirenberg inequality. Since  $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$ , however, it is important to make a distinction between  $B_R(z_0)$  and  $\mathbb{B}_R(z_0)$ .

We denote

$$M_R^\pm := \operatorname{ess\,sup}_{B_R(z_0)} u^\pm. \quad (5.55)$$

Let  $k \equiv k(R)$  be defined as in (5.5). Therefore, we now define  $w^\pm$  on  $\mathbb{B}_{4R}(z_0)$  by

$$w^\pm(z) := k + \begin{cases} -u^\pm(z) + M_{4R}^\pm, & z \in \mathbb{B}_{4R}(z_0) \cap B_{4R}(z_0), \\ +M_{4R}^\pm, & z \in \mathbb{B}_{4R}(z_0) \setminus B_{4R}(z_0). \end{cases} \quad (5.56)$$

As in the case of points in  $\Gamma_0$ , we may assume without loss of generality that (5.8) and (5.9) hold. From (5.53), we notice that  $M_{4R} \geq 0$  and  $m_{4R} \leq 0$ , and so it follows that  $M_{4R} = M_{4R}^+$  and  $m_{4R} = -M_{4R}^-$ . Therefore, assumption (5.8) becomes

$$M_{4R}^+ \neq 0 \quad \text{or} \quad M_{4R}^- \neq 0.$$

If  $M_{4R}^- = 0$ , then  $u = u^+$  on  $B_{4R}(z_0)$ , and it suffices to continue the following argument only for  $u^+$ . The same remark applies to  $M_{4R}^+ = 0$ . Thus, we may assume without loss of generality that

$$M_{4R}^+ \neq 0 \quad \text{and} \quad M_{4R}^- \neq 0. \quad (5.57)$$

Let  $\alpha < -1$ , and let  $\eta$  be a smooth cutoff function such that  $\text{supp } \eta \subseteq \mathbb{B}_{4R}(z_0)$ . We now define

$$v^\pm := \eta^2 \left[ (w^\pm)^\alpha - (k + M_{4R}^\pm)^\alpha \right]. \quad (5.58)$$

We notice that  $v^\pm$  is a well-defined function, for any choice of  $\alpha \in \mathbb{R}$ , by (5.57) and (5.9). By Lemma A.4,  $v^\pm \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  is a valid test function in (1.14). We observe that the function  $w^\pm$  obeys

$$k \leq w^\pm \leq k + M_{4R}^\pm \quad \text{on } \mathbb{B}_{4R}(z_0),$$

and, because  $\alpha$  is non-positive, we also have

$$k^\alpha \geq (w^\pm)^\alpha \geq (k + M_{4R}^\pm)^\alpha \quad \text{on } \mathbb{B}_{4R}(z_0).$$

These inequalities are important in deriving the analogues of the energy estimates in the proof of Theorem 1.11 for points in  $\Gamma_0$ . Steps 1 and 2 in the proof of Theorem 1.11 for points in  $\Gamma_0$  apply to our current choice of  $w^\pm$  for points in  $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ , with the only exception that we now define  $I(N)$  by

$$I(N) := \left( \int_{\mathbb{B}_{RN}(z_0)} |w^\pm|^{p_N} y^{\beta-1} dx dy \right)^{1/p_N}.$$

Therefore, we obtain the analogue of (5.23),

$$\text{ess inf}_{\mathbb{B}_R} w^\pm = I(-\infty) \geq C \left( \frac{1}{|\mathbb{B}_{2R}(z_0)|^{\beta-1}} \int_{\mathbb{B}_{2R}(z_0)} |w^\pm|^{p_0} y^{\beta-1} dx dy \right)^{1/p_0},$$

which implies

$$\text{ess inf}_{\mathbb{B}_R} w^\pm \geq C \left( \frac{1}{|\mathbb{B}_{2R}(z_0)|^{\beta-1}} \int_{\mathbb{B}_{2R}(z_0) \setminus B_{2R}(z_0)} |w^\pm|^{p_0} y^{\beta-1} dx dy \right)^{1/p_0}, \quad (5.59)$$

where  $p_0$  is a negative power and  $C$  is a positive constant depending on the coefficients of the Heston operator,  $n$ ,  $s$  and  $K$ . Notice that in the preceding second inequality, we used the fact that

$$|\mathbb{B}_R(z_0) \setminus B_R(z_0)|_{\beta-1} \neq 0. \quad (5.60)$$

Condition (5.60) is implied by (4.2), which follows from the exterior cone condition on  $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ , by (4.2). We claim that (5.56) implies

$$w^\pm = k + M_{4R}^\pm \geq M_{4R}^\pm \quad \text{on } \mathbb{B}_{2R}(z_0) \setminus B_{2R}(z_0). \quad (5.61)$$

Indeed, to see this we recall from (2.3) and (1.20) that  $\mathbb{B}_R(z_0)$  denotes the ball in the  $d$ -metric relative to the half-space,  $\mathbb{H}$ , while  $B_R(z_0)$  denotes the ball in the  $d$ -metric relative to the domain,  $\mathcal{O}$ . From the definition (5.56) of  $w^\pm$  we have

$$w^\pm = k + M_{4R}^\pm \geq M_{4R}^\pm \quad \text{on } \mathbb{B}_{4R}(z_0) \setminus B_{4R}(z_0).$$

and so (5.61) will follow from

$$\mathbb{B}_{2R}(z_0) \setminus B_{2R}(z_0) \subset \mathbb{B}_{4R}(z_0) \setminus B_{4R}(z_0). \quad (5.62)$$

But for any radius  $r > 0$ , we have

$$\mathbb{B}_r(z_0) \setminus B_r(z_0) = \{z \in \mathbb{B}_r(z_0) : z \notin \mathcal{O}\},$$

because  $B_r(z_0) = \mathbb{B}_r(z_0) \cap \mathcal{O}$ , and this gives (5.62). Thus, (5.61) holds as claimed.

Next, we claim that (5.56) implies

$$\operatorname{ess\,inf}_{\mathbb{B}_R(z_0)} w^\pm = k - M_R^\pm + M_{4R}^\pm. \quad (5.63)$$

To see this, observe that (5.56) gives

$$w^\pm = \begin{cases} k - u^\pm + M_{4R}^\pm & \text{on } B_R(z_0), \\ k + M_{4R}^\pm & \text{on } \mathbb{B}_R(z_0) \setminus B_R(z_0), \end{cases}$$

and  $k - u^\pm + M_{4R}^\pm \leq k + M_{4R}^\pm$  a.e. on  $B_R(z_0)$  since the  $u^\pm$  are non-negative. Therefore,

$$\begin{aligned} \operatorname{ess\,inf}_{\mathbb{B}_R(z_0)} w^\pm &= \operatorname{ess\,inf}_{B_R(z_0)} w^\pm \\ &= k + \operatorname{ess\,inf}_{B_R(z_0)} (-u^\pm) + M_{4R}^\pm \\ &= k - \operatorname{ess\,sup}_{B_R(z_0)} u^\pm + M_{4R}^\pm \\ &= k - M_R^\pm + M_{4R}^\pm, \quad (\text{by (5.55)}) \end{aligned}$$

which is (5.63) for  $w^\pm$ , as claimed.

Using (5.63) on the left-hand-side of (5.59) and (5.61) on the right-hand-side of (5.59), we obtain

$$\begin{aligned} k(R) - M_R^\pm + M_{4R}^\pm &\geq C \left( \frac{|\mathbb{B}_{2R}(z_0) \setminus B_{2R}(z_0)|_{\beta-1}}{|\mathbb{B}_{2R}(z_0)|_{\beta-1}} \right)^{1/p_0} (M_{4R}^\pm) \\ &\geq CM_{4R}^\pm, \end{aligned}$$

that is,

$$k(R) - M_R^\pm + M_{4R}^\pm \geq CM_{4R}^\pm. \quad (5.64)$$

Indeed, (5.64) follows because  $p_0 < 0$  and

$$\frac{|\mathbb{B}_{2R}(z_0)|_{\beta-1}}{|\mathbb{B}_{2R}(z_0) \setminus B_{2R}(z_0)|_{\beta-1}} \geq 1.$$

We rewrite (5.64), using  $\operatorname{osc}_{B_R(z_0)} u^\pm = M_R^\pm$ , as

$$\operatorname{osc}_{B_R(z_0)} u^\pm \leq C \operatorname{osc}_{B_{4R}(z_0)} u^\pm + k(R),$$

where  $C \in (0, 1)$  is a constant independent of  $R$ . Just as in the proof of Theorem 1.11 for the case of points in  $\Gamma_0$ , we can apply [28, Lemma 8.23] to conclude that (5.54) holds for  $u^\pm$  for positive constants  $C$ , depending on the coefficients of the Heston operator,  $n$ ,  $s$ , the cone  $K$ ,  $\bar{R}_0$ ,  $\|f\|_{L^s(B_{\bar{R}_0}(z_0), y^{\beta-1})}$  and  $\|u\|_{L^\infty(B_{\bar{R}_0}(z_0))}$ , and  $\alpha_0 \in (0, 1)$ , depending on  $s$ ,  $n$  and  $\beta$ , which implies that (1.27) holds for  $u$ , for possibly a different constant  $C$  with the same dependencies as before.

To establish (1.28), we proceed as in the proof of Theorem 1.11 for the case of points in  $\Gamma_0$ . In order to adapt the argument for the case of points in  $\Gamma_0$  to points in  $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ , we need analogues of the inequalities (5.36) to hold in a neighborhood in  $\mathcal{O}$  of  $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$ . Given these analogues

of the inequalities (5.36), we can apply the same argument as used in the Step 5 of the proof of Theorem 1.11 for the case of points in  $\Gamma_0$ , but instead of applying [28, Theorem 8.22], we now apply [28, Theorem 8.27]. As before, we assume (5.35) holds.

Without loss of generality, we may assume  $z_0 = (0, 0)$ . Let  $z_1 = (x_1, 0)$ ,  $z_2 = (x_2, 0)$ ,  $z_3 = (x, y)$  and  $z_4 = (x, 0)$  be points in  $\bar{B}_R(z_0)$ . We may assume  $x_2 \geq x_1$  and  $x, x_1, x_2 \geq 0$ . We claim that the following analogues of the inequalities (5.36) (for points  $z_0 \in \Gamma_0$ ) hold for points  $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$ ,

$$\begin{aligned} |u(z_1) - u(z_2)| &\leq C_3 d(z_1, z_2)^{\alpha_3}, \\ |u(z_3) - u(z_4)| &\leq C_3 d(z_3, z_4)^{\alpha_3}, \end{aligned} \quad (5.65)$$

for some positive constant  $C_3$  and  $\alpha_3 \in (0, 1)$  satisfying the same dependency conditions as in the statement of Theorem 1.11. For the *first* inequality in (5.65), we consider two cases.

**Case 1** (Points  $z_1, z_2 \in \bar{B}_R(z_0)$  obeying (5.66)). If

$$d(z_1, z_2) \geq \frac{1}{8} \max \{d(z_1, z_0), d(z_2, z_0)\}, \quad (5.66)$$

then we have

$$\begin{aligned} |u(z_1) - u(z_2)| &\leq |u(z_1) - u(z_0)| + |u(z_2) - u(z_0)| \\ &\leq C d(z_1, z_0)^{\alpha_0} + C d(z_2, z_0)^{\alpha_0} \quad (\text{by (1.27)}) \\ &\leq C d(z_1, z_2)^{\alpha_0} \quad (\text{by (5.66)}), \end{aligned}$$

and so the first inequality in (5.65) holds in this case.

**Case 2** (Points  $z_1, z_2 \in \bar{B}_R(z_0)$  obeying (5.67)). If

$$d(z_1, z_2) \leq \frac{1}{8} \max \{d(z_1, z_0), d(z_2, z_0)\}, \quad (5.67)$$

then, we apply (5.52) on the ball  $B_{\tilde{R}}(z_2)$  with  $\tilde{R} = d(z_1, z_2)$ .

Combining the preceding two cases, we obtain the first inequality in (5.65).

Next, we consider the *second* inequality in (5.65). By (5.37), we have

$$d(z_3, z_4) = \sqrt{y/2} \quad \text{and} \quad d(z_4, z_0) = \sqrt{x}. \quad (5.68)$$

As in the proof of the first inequality in (5.65), we consider two possible cases.

**Case 1** (Points  $z_3, z_4 \in \bar{B}_R(z_0)$  obeying (5.69)). If

$$x \geq 32y, \quad (5.69)$$

then, by (5.68), we have  $d(z_3, z_4) \leq 1/8 d(z_4, z_0)$ . We may apply (5.52) on the ball  $B_{\tilde{R}}(z_4)$  with  $\tilde{R} = d(z_3, z_4)$ , and we obtain the second inequality in (5.65).

**Case 2** (Points  $z_3, z_4 \in \bar{B}_R(z_0)$  obeying (5.70)). If

$$x < 32y, \quad (5.70)$$

then we have  $d(z_4, z_0) \leq 8d(z_3, z_4)$ . Also, a direct calculation gives us  $d(z_3, z_0) \leq C d(z_3, z_4)$ , for some positive constant  $C$ . By (1.27), we obtain

$$\begin{aligned} |u(z_3) - u(z_4)| &\leq |u(z_3) - u(z_0)| + |u(z_4) - u(z_0)| \\ &\leq C d(z_3, z_0)^{\alpha_0} + C d(z_4, z_0)^{\alpha_0} \\ &\leq 2C d(z_3, z_4)^{\alpha_0}, \end{aligned}$$

and we obtain the second inequality in (5.65).

The proof of (5.65) is complete. We may now conclude, by applying the same argument as in Step 5 of the proof of Theorem 1.11 for the case of points in  $\Gamma_0$ , that for any  $z_1, z_2 \in \bar{B}_R(z_0)$ , where  $R$  satisfies (5.1), (5.2) and (5.35), we have

$$|u(z_1) - u(z_2)| \leq C_4 d(z_1, z_2)^{\alpha_4}, \quad (5.71)$$

where  $C_4$  and  $\alpha_4$  are constants satisfying the dependencies stated in Theorem 1.11 for  $C_1$  and  $\alpha_1$ , respectively.  $\square$

*Completion of proof of Theorem 1.11.* It remains to complete the proof of inequality (1.28). Notice that inequalities (5.52) and (5.71) are slightly weaker than (1.28), because they apply to points  $z_1, z_2 \in \bar{B}_R(z_0)$ , where  $R$  is required to satisfy assumptions (5.1), (5.2) and (5.35), instead of assuming  $R = \bar{R}$  as in (1.28). To obtain (1.28), all we need to notice is that (5.52) and (5.71) imply that  $u$  is  $C_s^{\alpha_5}$ -Hölder continuous on  $B_{\bar{R}}(z_0) \cap (\bar{\Gamma}_0 \times [0, R_1])$ , for some positive constant  $R_1$  and for some  $\alpha_5 \in (0, 1)$  satisfying the same dependencies as  $\alpha_1$  in Theorem 1.11. The constant  $R_1$  is chosen small enough that it satisfies assumptions (5.1), (5.2) and (5.35), when  $R$  is replaced by  $R_1$ . On  $\bar{B}_{\bar{R}_0}(z_0) \cap \{y \geq R_1\}$ , the operator  $A$  is a uniformly elliptic with bounded coefficients, so [28, Theorem 8.29] applies and, for an  $\alpha_3 \in (0, 1)$ , we see that  $u$  is  $C^{\alpha_3}$ -Hölder continuous with respect to the Euclidean metric on  $\bar{B}_{\bar{R}}(z_0) \cap \{y \geq R_1\}$ . Because the Euclidean and the Koch metric,  $d$ , are equivalent on  $\bar{B}_{\bar{R}}(z_0) \cap \{y \geq R_1\}$ , we see that  $u$  is  $C_s^{\alpha_3}$ -Hölder continuous with respect to the Koch metric,  $d$ , on  $\bar{B}_{\bar{R}}(z_0) \cap \{y \geq R_1\}$ . The Hölder exponent,  $\alpha_3$ , depends on the coefficients of the Heston operator,  $A$ , together with  $n$ ,  $s$ ,  $\bar{R}_0$  and on the cone  $K$  defining the uniform exterior cone condition along  $\bar{\Gamma}_1 \cap \bar{B}_{\bar{R}_0}(z_0)$ , if  $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$ . Therefore, we obtain inequality (1.28) with an  $\alpha_1 \in (0, 1)$  and a constant  $C_1 > 0$  depending on the coefficients of the Heston operator,  $A$ , together with  $\bar{R}_0$ ,  $s$ ,  $n$ , and  $K$ . The constant  $C_1$  depends in addition on

$$\|f\|_{L^s(B_{\bar{R}_0}(z_0), y^{\beta-1})} \quad \text{and} \quad \|u\|_{L^\infty(B_{\bar{R}_0}(z_0))}.$$

This completes the proof of Theorem 1.11.  $\square$

### 5.3. Hölder continuity of solutions to the variational equation on a neighborhood of the portion of the boundary where $A$ is degenerate.

We conclude this section with the

*Proof of Corollary 1.12.* We let  $R_1$  be small enough so that it is less than the values of the constants  $\bar{R}/2$  appearing in Theorem 1.7 and in Theorem 1.11, applied with  $\bar{R}_0 := \delta$ . In addition, we choose  $R_1$  such that

$$B_{R_1}(z_0) \subset B_{\delta/2}(z_0), \quad \forall z_0 \in \bar{\Gamma}_0.$$

Recall definitions (1.20) and (2.3) for balls defined by the Koch metric  $d$ , and definitions (2.5) and (2.4) for the balls defined by the Euclidean metric. By choosing  $R_2 := R_1^2/2000$ , we see that (2.7) implies  $E_{R_2}(z_0) \subset B_{R_1}(z_0)$ , for all  $z_0 \in \partial\mathbb{H}$ , and clearly also that

$$E_{R_2}(z_0) \subset B_{R_1}(z_0) \subset B_{\delta/2}(z_0), \quad \forall z_0 \in \bar{\Gamma}_0. \quad (5.72)$$

By (5.72), we can find a sequence of points  $\{z_0^n\}_{n \in \mathbb{N}} \subset \Gamma_0$  such that

$$\mathcal{O}_{R_2/2} \subset \bigcup_{n \in \mathbb{N}} E_{R_2}(z_0^n) \subset \bigcup_{n \in \mathbb{N}} B_{R_1}(z_0^n) \subset \bigcup_{z_0 \in \Gamma_0} B_{\delta/2}(z_0).$$

We choose  $\delta_1 := R_2/2$  and so we obtain

$$\mathcal{O}_{\delta_1} \subset \bigcup_{n \in \mathbb{N}} B_{R_1}(z_0^n) \subset \bigcup_{z_0 \in \Gamma_0} B_{\delta/2}(z_0). \quad (5.73)$$



By (1.30), we have  $f \in L^s(B_\delta(z_0^n), y^{\beta-1})$  and  $u \in L^2(B_\delta(z_0^n), y^{\beta-1})$ , for all  $n \in \mathbb{N}$ . Because the constant  $C$  in (1.21) does not depend on the point  $z_0 \in \bar{\Gamma}_0$ , we deduce from (5.73), by applying the estimate (1.21) on each of the balls  $B_{R_1}(z_0^n)$ ,  $n \in \mathbb{N}$ , that

$$\sup_{z_0 \in \Gamma_0} \|u\|_{L^\infty(B_{R_1}(z_0))} \leq C \left( \sup_{z_0 \in \Gamma_0} \|f\|_{L^s(B_\delta(z_0))} + \sup_{z_0 \in \Gamma_0} \|u\|_{L^2(B_\delta(z_0))} \right), \quad (5.74)$$

for some constant  $C$  depending only on  $n$ ,  $s$ ,  $K$  and the coefficients of the Heston operator. This yields  $u \in L^\infty(\mathcal{O}_{\delta_1})$ . By Theorem 1.11, we know that  $u$  is continuous on  $\bar{\mathcal{O}}_{\delta_1}$ , from which it follows that  $u \in C_{\text{loc}}(\bar{\mathcal{O}}_{\delta_1})$ .

We show next that, for some constant  $C_2$ , satisfying the same dependencies as the constant  $C_1$  in the statement of Theorem 1.11, we have

$$|u(z_1) - u(z_2)| \leq C_2 d(z_1, z_2)^{\alpha_1}, \quad \forall z_1, z_2 \in \bar{\mathcal{O}}_{\delta_1}, \quad (5.75)$$

from which it follows that  $u \in C_s^{\alpha_1}(\bar{\mathcal{O}}_{\delta_1})$ . The Hölder exponent  $\alpha_1 \in (0, 1)$  is as in Theorem 1.11. To prove (5.75), we consider two cases. We fix  $z_1, z_2 \in \mathcal{O}_{\delta_1}$ .

**Case 1** ( $d(z_1, z_2) < R_1/2$ ). First, we assume that  $d(z_1, z_2) < R_1/2$ . Because  $R_1$  was chosen such that  $R_1 < \bar{R}/2$ , where  $\bar{R}$  is the constant appearing in Theorem 1.11, we clearly have that  $d(z_1, z_2) < \bar{R}/2$ . By (5.73), we see that  $z_1 \in B_{R_1}(z_0^n)$  for some  $n \in \mathbb{N}$ , and so  $d(z_1, z_0^n) < R_1 < \bar{R}/2$ , which implies that  $d(z_2, z_0^n) < \bar{R}$ . We obtain that  $z_1, z_2 \in B_{\bar{R}}(z_0^n)$ , with  $z_0^n \in \Gamma_0$ , so we may apply inequality (1.28) on  $B_{\bar{R}}(z_0^n)$  to conclude that (5.75) holds for points  $z_1, z_2 \in \mathcal{O}_{\delta_1}$  with  $d(z_1, z_2) < R_1/2$ .

**Case 2** ( $d(z_1, z_2) \geq R_1/2$ ). If  $z_1, z_2 \in \mathcal{O}_{\delta_1}$  and  $d(z_1, z_2) \geq R_1/2$ , then using the fact that  $u \in L^\infty(\mathcal{O}_{\delta_1})$ , we have

$$|u(z_1) - u(z_2)| \leq 2\|u\|_{L^\infty(\mathcal{O}_{\delta_1})} \frac{d(z_1, z_2)^{\alpha_1}}{(R_1/2)^{\alpha_1}}.$$

By combining the preceding two cases, we see that (5.75) holds for all points  $z_1, z_2 \in \mathcal{O}_{\delta_1}$ . Moreover, using also (5.74), we see that the Hölder norm of  $u$  on  $\bar{\mathcal{O}}_{\delta_1}$  depends on the coefficients of the Heston operator,  $A$ , together with  $n$ ,  $s$ ,  $K$ ,  $\delta$  and the supremum bounds in (1.30). This completes the proof of Corollary 1.12.  $\square$

**Remark 5.2** (Hölder continuity up to  $\Gamma_0$  and Sobolev embeddings). Hölder continuity of solutions does not follow by an embedding theorem for Sobolev weighted spaces, analogous to [28, Corollary 7.11], not even for functions  $u \in H^2(\mathcal{O}, \mathfrak{w})$ . For example, for any  $\beta > 2$ , let  $q \in (0, (\beta - 2)/2)$  and

$$u(x, y) = y^{-q}, \quad \forall (x, y) \in \mathcal{O}.$$

Then,  $u \in H^2(\mathcal{O}, \mathfrak{w})$ , but  $u \notin C_s^\alpha(\mathcal{O})$ , for any  $\alpha \in [0, 1]$ , since, a fortiori,  $u \notin C(\mathcal{O} \cup \Gamma_0)$ .

## 6. HÖLDER CONTINUITY FOR SOLUTIONS TO THE VARIATIONAL INEQUALITY

In this section, we use the penalization method and a priori estimates for solutions to the penalized equation derived in [15] together with Theorem 1.11 to prove local Hölder continuity on a neighborhood of  $\bar{\Gamma}_0$  in  $\bar{\mathcal{O}}$  for solutions  $u$  to the variational inequality (1.2) (Theorem 1.16).

**6.1. Reduction to a finite-height domain.** If  $\text{height}(\mathcal{O}) = \infty$ , we shall need to avail of the second condition in (1.36) to enable cutting off the solution and use localization to reduce to the case of a finite-height domain. Let  $\mathcal{U} \subseteq \mathcal{O}$  be an open subset. Suppose we are given an open subset  $\mathcal{V} \subset \mathcal{U}$  with  $\bar{\mathcal{V}} \setminus \partial\mathcal{O} \subset \mathcal{U}$  and

$$\text{dist}(\mathcal{O} \cap \partial\mathcal{V}, \mathcal{O} \cap \partial\mathcal{U}) > 0. \quad (6.1)$$

Let  $\zeta \in C^\infty(\bar{\mathbb{H}})$  be a cutoff function such that  $0 \leq \zeta \leq 1$  on  $\mathbb{H}$ ,  $\zeta = 1$  on  $\mathcal{V}$ ,  $\zeta > 0$  on  $\mathcal{U}$ , and  $\zeta = 0$  on  $\mathcal{O} \setminus \mathcal{U}$ . By (6.1) and construction of  $\zeta$ , there is a positive constant,  $C_0$ , depending only on  $\text{dist}(\mathcal{O} \cap \partial\mathcal{V}, \mathcal{O} \cap \partial\mathcal{U})$  such that

$$\|\zeta\|_{C^2(\mathbb{H})} \leq C_0. \quad (6.2)$$

We obtain  $\zeta\psi \in H^1(\mathcal{U}, \mathfrak{w})$  by (6.2) and the fact that  $\psi \in H^1(\mathcal{O}, \mathfrak{w})$ . Because  $\zeta = 0$  on  $\partial\mathcal{U} \setminus \partial\mathcal{O}$  and  $\psi \leq 0$  on  $\Gamma_1 = \partial\mathcal{O} \setminus \bar{\Gamma}_0$  (trace sense), then  $\zeta\psi \leq 0$  on  $\partial\mathcal{U} \setminus \bar{\Gamma}_0$  (trace sense). Similarly, as  $\zeta = 0$  on  $\partial\mathcal{U} \setminus \partial\mathcal{O}$  and  $u = 0$  on  $\partial\mathcal{O} \setminus \bar{\Gamma}_0$  (trace sense), then  $\zeta u = 0$  on  $\partial\mathcal{U} \setminus \bar{\Gamma}_0$  (trace sense) and therefore

$$\zeta u \in H_0^1(\mathcal{U} \cup \Gamma_0, \mathfrak{w}) \quad (6.3)$$

by [15, Lemma A.32].

**Lemma 6.1** (Localization of solutions to variational inequalities). [15, Claim 6.16] *If  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  is a solution to (1.2) with obstacle function,  $\psi \in H^1(\mathcal{O}, \mathfrak{w})$  with  $\psi^+ \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ , and source function,  $f \in L^2(\mathcal{O}, \mathfrak{w})$ , then  $\zeta u \in H_0^1(\mathcal{U} \cup \Gamma_0, \mathfrak{w})$  is a solution to the variational inequality (1.2) on  $\mathcal{U}$  with obstacle function,  $\zeta\psi \in H^1(\mathcal{U}, \mathfrak{w})$  with  $\zeta\psi^+ \in H_0^1(\mathcal{U} \cup \Gamma_0, \mathfrak{w})$ , and source function,*

$$f_\zeta := \zeta f + [A, \zeta]u \in L^2(\mathcal{U}, \mathfrak{w}). \quad (6.4)$$

**Remark 6.2** (Reduction to the case of a finite-height domain). In order to reduce the case of a domain  $\mathcal{O} \subseteq \mathbb{H}$  with  $\text{height}(\mathcal{O}) = \infty$  to the case of a finite-height domain  $\mathcal{O} \subseteq \mathbb{R} \times (0, \delta)$ , for some  $\delta > 0$ , we can apply Lemma 6.1 to the choice

$$\zeta = \begin{cases} 1 & \text{on } \mathbb{R} \times (-\infty, \delta/2], \\ 0 & \text{on } \mathbb{R} \times [3\delta/4, \infty), \end{cases} \quad (6.5)$$

given by  $\zeta(x, y) = \chi(y/\delta)$ ,  $(x, y) \in \mathbb{R}^2$ , where  $\chi \in C^\infty(\bar{\mathbb{R}})$  is a cutoff function with  $0 \leq \chi \leq 1$  on  $\mathbb{R}$ ,  $\chi(t) = 1$ ,  $t \leq 1/2$ , and  $\chi(t) = 0$ ,  $t \geq 3/4$ . Observe that  $\text{supp}[A, \zeta]u \subset \mathbb{R} \times [\delta/2, 3\delta/4]$  in (6.4) and that, because  $u$  obeys (1.36), we obtain

$$f_\zeta \in L^2(\mathcal{O}_\delta, \mathfrak{w}) \cap L^\infty(\mathcal{O}_\delta),$$

and thus  $f_\zeta$  obeys (1.33), while

$$\zeta u = u \quad \text{on } \mathcal{O}_{\delta/2}, \quad (6.6)$$

with  $\mathcal{O}_\delta$  as in Hypothesis 1.15.

**6.2. Proof of Hölder continuity up to  $\bar{\Gamma}_0$  for solutions to the variational inequality.** By Remark 6.2, we may assume without loss of generality for the remainder of this section that  $\mathcal{O}$  has finite height,

$$\mathcal{O} \subseteq \mathbb{R} \times (0, \delta), \quad (6.7)$$

where  $\delta > 0$  is as in Hypothesis 1.15, with source function (relabelled if necessary),  $f$ , obeying (1.33) and obtain the desired Hölder continuity for  $u$  along the subdomain  $\mathcal{O}_{\delta/2}$  via (6.6).

We shall prove Theorem 1.16 using the *method of penalization*, following the pattern in [15], by first deriving an  $L^\infty$  bound on a *penalization term*,  $\beta_\varepsilon(u_\varepsilon - \psi)$  in the semilinear *penalized equation* (6.10) corresponding to the variational inequality (1.32), which is uniform with respect

to  $\varepsilon \in (0, \varepsilon_0]$ , for some sufficiently small positive constant  $\varepsilon_0$ . We then appeal to Theorem 1.11 to conclude that the family of functions  $\{u_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0]}$  solving the penalized equation is  $C^{\alpha_0}$ -continuous up to  $\bar{\Gamma}_0$  and hence, by passing to a subsequence and taking limits, via the convergence results in [15], that the same is true for a solution,  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ , to (1.32). Following [15, Equations (3.1) & (3.2)], we denote

$$a_\lambda(u, v) := a(u, v) + \lambda((1 + y)u, v)_{L^2(\mathcal{O}, \mathfrak{w})}, \quad \forall u, v \in H^1(\mathcal{O}, \mathfrak{w}), \quad (6.8)$$

$$A_\lambda := A + \lambda(1 + y), \quad (6.9)$$

where  $\lambda \geq 0$  and, as usual,  $a(u, v)$  is given by (1.14) and  $A$  by (1.4).

**Lemma 6.3** (Uniform bound on the penalization term). *Let  $f \in L^2(\mathcal{O}, \mathfrak{w}) \cap L^\infty(\mathcal{O})$  and  $\psi \in H^2(\mathcal{O}, \mathfrak{w}) \cap L^\infty(\mathcal{O})$  obey (1.31). For  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  obeying  $u \geq \psi$  a.e. on  $\mathcal{O}$  and  $\lambda \geq 0$ , and  $\varepsilon > 0$ , let  $u_\varepsilon \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}) \cap L^\infty(\mathcal{O})$  be a solution to the penalized equation,*

$$a_\lambda(u_\varepsilon, v) + (\beta_\varepsilon(u_\varepsilon - \psi), v)_{L^2(\mathcal{O}, \mathfrak{w})} = (f_\lambda, v)_{L^2(\mathcal{O}, \mathfrak{w})}, \quad \forall v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}), \quad (6.10)$$

defined by the penalization function,

$$\beta_\varepsilon(t) := -\frac{1}{\varepsilon}t^-, \quad t \in \mathbb{R}, \quad (6.11)$$

where  $t^- := -\min\{t, 0\}$ , and<sup>1</sup>

$$f_\lambda := f + \lambda(1 + y)u \in L^2(\mathcal{O}, \mathfrak{w}). \quad (6.12)$$

If  $\lambda + r > 0$ , there is a positive constant  $\varepsilon_0$ , depending only on  $\lambda$  and the coefficients of the Heston operator,  $A$  in (1.4), such that

$$\|\beta_\varepsilon(u_\varepsilon - \psi)\|_{L^\infty(\mathcal{O})} \leq 2 \operatorname{ess\,sup}_{\mathcal{O}} (A\psi - f)^+, \quad \forall \varepsilon \in (0, \varepsilon_0]. \quad (6.13)$$

*Proof.* We adapt an argument used in the proof of [50, Theorem 4.38]. Integration by parts [15, Lemma 2.23] with  $\psi \in H^2(\mathcal{O}, \mathfrak{w})$  and  $v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  yields

$$a_\lambda(\psi, v) = (A_\lambda \psi, v)_{L^2(\mathcal{O}, \mathfrak{w})}. \quad (6.14)$$

Since  $u_\varepsilon \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  and  $\psi^+ \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ , it follows that  $\beta_\varepsilon(u_\varepsilon - \psi) \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  by the proof of [15, Lemma A.34]. In order to use  $\beta_\varepsilon(u_\varepsilon - \psi)$  to construct suitable test functions, we need the

**Claim 6.4** (Boundedness of the penalization term). *The penalization term,  $\beta_\varepsilon(u_\varepsilon - \psi)$ , is in  $L^\infty(\mathcal{O})$ .*

*Proof of Claim 6.4.* Since  $\beta_\varepsilon(u - \psi) \leq 0$  a.e. on  $\mathcal{O}$ , we have

$$a_\lambda(u_\varepsilon, v) = (f, v)_{L^2(\mathcal{O}, \mathfrak{w})} - (\beta_\varepsilon(u - \psi), v)_{L^2(\mathcal{O}, \mathfrak{w})} \geq (f, v)_{L^2(\mathcal{O}, \mathfrak{w})},$$

for all  $v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  with  $v \geq 0$  a.e. on  $\mathcal{O}$ , and so the weak maximum principle [24, Proposition 6.10 & Theorem 8.15] for  $a_\lambda$  given by (6.8) implies that

$$u_\varepsilon \geq 0 \wedge \frac{1}{\lambda + r} \operatorname{ess\,inf}_{\mathcal{O}} f \quad \text{a.e. on } \mathcal{O},$$

---

<sup>1</sup>Not to be confused with  $f_\zeta$  as defined in equation (6.4).

where  $x \wedge y := \min\{x, y\}$ ,  $\forall x, y \in \mathbb{R}$ , and hence

$$\begin{aligned} (u_\varepsilon - \psi)^- &= (\psi - u_\varepsilon)^+ \leq \left( \operatorname{ess\,sup}_{\mathcal{O}} \psi - \operatorname{ess\,inf}_{\mathcal{O}} u_\varepsilon \right)^+ \\ &\leq \left( \operatorname{ess\,sup}_{\mathcal{O}} \psi - 0 \wedge \frac{1}{\lambda + r} \operatorname{ess\,inf}_{\mathcal{O}} f \right)^+ \quad \text{a.e. on } \mathcal{O}. \end{aligned}$$

Since  $(u_\varepsilon - \psi)^- \geq 0$  and  $f, \psi \in L^\infty(\mathcal{O})$  by hypothesis, it follows that  $(u_\varepsilon - \psi)^- \in L^\infty(\mathcal{O})$  and thus  $\beta_\varepsilon(u_\varepsilon - \psi) \in L^\infty(\mathcal{O})$ , as desired.  $\square$

If  $F(t) := t^{q-1}$ , for  $q > 2$ , and  $F'(t) = (q-1)t^{q-2}$ , for  $t \in \mathbb{R}$ , then the proofs of [28, Lemmas 7.5 & 7.6 and Theorem 7.8] (see [15, Lemma A.34] and its proof) and the fact that  $\beta_\varepsilon(u_\varepsilon - \psi) \in L^\infty(\mathcal{O})$  by Claim 6.4 show that

$$v := |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1} \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}). \quad (6.15)$$

By subtracting (6.14) from (6.10) and choosing  $v$  as in (6.15), we obtain

$$\begin{aligned} a_\lambda(u_\varepsilon - \psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1}) &+ (\beta_\varepsilon(u_\varepsilon - \psi), |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1})_{L^2(\mathcal{O}, \mathfrak{w})} \\ &= (f_\lambda - A_\lambda \psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1})_{L^2(\mathcal{O}, \mathfrak{w})}. \end{aligned} \quad (6.16)$$

Since  $u \geq \psi$  a.e. on  $\mathcal{O}$  by hypothesis, the term on the right-hand side of equation (6.16) obeys

$$(f_\lambda - A_\lambda \psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1})_{L^2(\mathcal{O}, \mathfrak{w})} \geq (f - A\psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1})_{L^2(\mathcal{O}, \mathfrak{w})}, \quad (6.17)$$

since  $f_\lambda - A_\lambda \psi = f + \lambda(1+y)(u - \psi) - A\psi \geq f - A\psi$  a.e. on  $\mathcal{O}$  by (6.9) and (6.12). Notice that

$$(\beta_\varepsilon(u_\varepsilon - \psi), |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1})_{L^2(\mathcal{O}, \mathfrak{w})} = - \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q \mathfrak{w} \, dx \, dy \quad (6.18)$$

and so (6.16), (6.17), and (6.18) yield

$$\begin{aligned} a_\lambda(u_\varepsilon - \psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1}) &- \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q \mathfrak{w} \, dx \, dy \\ &\geq (f - A\psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1})_{L^2(\mathcal{O}, \mathfrak{w})}. \end{aligned} \quad (6.19)$$

Observe that (6.15) gives

$$v_x = -(q-1)|\beta_\varepsilon(u_\varepsilon - \psi)|^{q-2} \beta'_\varepsilon(u_\varepsilon - \psi)(u_\varepsilon - \psi)_x,$$

and similarly for  $v_y$ . By a straightforward calculation using the expression (6.8) for  $a_\lambda(u, v)$ , we find that

$$\begin{aligned} &a_\lambda(u_\varepsilon - \psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1}) \\ &= \frac{1}{2} \int_{\mathcal{O}} [(u_\varepsilon - \psi)_x^2 + 2\rho\sigma(u_\varepsilon - \psi)_x(u_\varepsilon - \psi)_y + \sigma^2(u_\varepsilon - \psi)_y^2] \\ &\quad \times (q-1)|\beta_\varepsilon(u_\varepsilon - \psi)|^{q-2} \beta'_\varepsilon(u_\varepsilon - \psi) y \mathfrak{w} \, dx \, dy \\ &\quad - \frac{\gamma}{2} \int_{\mathcal{O}} [(u_\varepsilon - \psi)_x + \rho\sigma(u_\varepsilon - \psi)_y] |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1} \operatorname{sign}(x) y \mathfrak{w} \, dx \, dy \\ &\quad - \frac{1}{2} \int_{\mathcal{O}} a_1(u_\varepsilon - \psi)_x |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1} y \mathfrak{w} \, dx \, dy \\ &\quad + \int_{\mathcal{O}} (r + \lambda(1+y)) (u_\varepsilon - \psi) |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1} y \mathfrak{w} \, dx \, dy. \end{aligned} \quad (6.20)$$

We write the sum of integrals on the right-hand side of (6.20) as  $I_1 + I_2 + I_3 + I_4$ . By the uniform ellipticity (1.16), we find that there exists a positive constant  $C_1$ , depending only on the coefficients of the Heston operator,  $A$  in (1.4), such that

$$I_1 \geq (q-1)C_1 \int_{\mathcal{O}} |\nabla(u_\varepsilon - \psi)|^2 \beta'_\varepsilon(u_\varepsilon - \psi) |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-2} y \mathfrak{w} \, dx \, dy,$$

noting that  $\beta'_\varepsilon(t) \geq 0$  a.e.  $t \in \mathbb{R}$ . Indeed, by (6.11) we have

$$\beta'_\varepsilon(t) = \frac{1}{\varepsilon} 1_{\{t \leq 0\}} \leq \frac{1}{\varepsilon} \quad \text{a.e. } t \in \mathbb{R},$$

and so the identity,

$$\nabla \beta_\varepsilon(u_\varepsilon - \psi) = \beta'_\varepsilon(u_\varepsilon - \psi) \nabla(u_\varepsilon - \psi) = \frac{1}{\varepsilon} 1_{\{t \leq 0\}} \nabla(u_\varepsilon - \psi) \quad \text{a.e. on } \mathcal{O}, \quad (6.21)$$

yields

$$\begin{aligned} |\nabla(u_\varepsilon - \psi)|^2 \beta'_\varepsilon(u_\varepsilon - \psi) &= \frac{1}{\varepsilon} |\nabla(u_\varepsilon - \psi)|^2 1_{\{t \leq 0\}} \\ &= \varepsilon |\nabla \beta_\varepsilon(u_\varepsilon - \psi)|^2 1_{\{t \leq 0\}} \\ &= \varepsilon |\nabla \beta_\varepsilon(u_\varepsilon - \psi)|^2 \quad \text{a.e. on } \mathcal{O}. \end{aligned}$$

Hence, by combining the preceding inequality and identity, we see that

$$I_1 \geq \varepsilon(q-1)C_1 \int_{\mathcal{O}} |\nabla \beta_\varepsilon(u_\varepsilon - \psi)|^2 |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-2} y \mathfrak{w} \, dx \, dy. \quad (6.22)$$

Using (6.21) and the fact that  $\beta_\varepsilon(t) 1_{\{t \leq 0\}} = \beta_\varepsilon(t)$ , we can write  $I_2$  in the form

$$\begin{aligned} I_2 &= -\varepsilon \frac{\gamma}{2} \int_{\mathcal{O}} \left[ (\beta_\varepsilon(u_\varepsilon - \psi))_x + \rho \sigma (\beta_\varepsilon(u_\varepsilon - \psi))_y \right] \\ &\quad \times |\beta_\varepsilon(u_\varepsilon - \psi)|^{(q-2)/2} |\beta_\varepsilon(u_\varepsilon - \psi)|^{q/2} \text{sign}(x) y \mathfrak{w} \, dx \, dy. \end{aligned}$$

Hence, there is a positive constant  $C_2$ , depending only on the coefficients of the Heston operator,  $A$  in (1.4), and  $\gamma$  (which in turn, by [15], can be assumed to depend only those coefficients), such that for any  $\eta > 0$ ,

$$|I_2| \leq \varepsilon \eta \int_{\mathcal{O}} |\nabla \beta_\varepsilon(u_\varepsilon - \psi)|^2 |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-2} y \mathfrak{w} \, dx \, dy + C_2 \frac{\varepsilon}{\eta} \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q y \mathfrak{w} \, dx \, dy. \quad (6.23)$$

Similarly, we obtain for  $I_3$ , for any  $\eta > 0$ ,

$$|I_3| \leq \varepsilon \eta \int_{\mathcal{O}} |\nabla \beta_\varepsilon(u_\varepsilon - \psi)|^2 |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-2} y \mathfrak{w} \, dx \, dy + C_3 \frac{\varepsilon}{\eta} \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q y \mathfrak{w} \, dx \, dy, \quad (6.24)$$

where  $C_3$  is a positive constant depending only on the coefficients of the Heston operator,  $A$  in (1.4). We can also estimate  $I_4$  by

$$|I_4| \leq \varepsilon C_4 \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q \mathfrak{w} \, dx \, dy, \quad (6.25)$$

where  $C_4$  is a positive constant depending only on  $\lambda$ , the coefficients of the Heston operator,  $A$  in (1.4), and the height of the domain  $\mathcal{O}$ . Substituting (6.22), (6.23), (6.24) and (6.25) in (6.20),

we obtain

$$\begin{aligned} & a_\lambda(u_\varepsilon - \psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1}) \\ & \geq -\varepsilon \left( \frac{C_2}{\eta} + \frac{C_3}{\eta} + C_4 \right) \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q \mathfrak{w} \, dx \, dy \\ & \quad + \varepsilon((q-1)C_1 - 2\eta) \int_{\mathcal{O}} |\nabla \beta_\varepsilon(u_\varepsilon - \psi)|^2 |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-2} y \mathfrak{w} \, dx \, dy. \end{aligned}$$

Choose  $\eta := C_1/2$  and, noting that  $q > 2$ , we have  $(q-1)C_1 - 2\eta \geq 0$  and thus

$$a_\lambda(u_\varepsilon - \psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1}) \geq -\varepsilon C \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q \mathfrak{w} \, dx \, dy, \quad (6.26)$$

where  $C := 2C_2/C_1 + 2C_3/C_1 + C_4$ . But (6.19) gives

$$\begin{aligned} & \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q \mathfrak{w} \, dx \, dy \\ & \leq -(f - A\psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1})_{L^2(\mathcal{O}, \mathfrak{w})} - a_\lambda(u_\varepsilon - \psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1}) \\ & \leq -(f - A\psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1})_{L^2(\mathcal{O}, \mathfrak{w})} + \varepsilon C \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q \mathfrak{w} \, dx \, dy \quad (\text{by (6.26)}) \end{aligned}$$

and thus,

$$\begin{aligned} (1 - \varepsilon C) \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q \mathfrak{w} \, dx \, dy & \leq (A\psi - f, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1})_{L^2(\mathcal{O}, \mathfrak{w})} \\ & \leq ((A\psi - f)^+, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1})_{L^2(\mathcal{O}, \mathfrak{w})}. \end{aligned}$$

Now choose  $\varepsilon_0 = 2/C$  and so  $(1 - \varepsilon C) \geq 1/2$ , for any  $0 < \varepsilon \leq \varepsilon_0$ . By applying the Hölder inequality on the right-hand side, we see that

$$\frac{1}{2} \|\beta_\varepsilon(u_\varepsilon - \psi)\|_{L^q(\mathcal{O}, \mathfrak{w})}^q \leq \|(A\psi - f)^+\|_{L^q(\mathcal{O}, \mathfrak{w})} \|\beta_\varepsilon(u_\varepsilon - \psi)\|_{L^q(\mathcal{O}, \mathfrak{w})}^{q-1}, \quad 0 < \varepsilon \leq \varepsilon_0,$$

and so

$$\frac{1}{2} \|\beta_\varepsilon(u_\varepsilon - \psi)\|_{L^q(\mathcal{O}, \mathfrak{w})} \leq \|(A\psi - f)^+\|_{L^q(\mathcal{O}, \mathfrak{w})}, \quad \forall q > 2,$$

which yields, by taking the limit as  $q \rightarrow \infty$  and applying Lemma A.6, the desired inequality (6.13).  $\square$

Solutions to (6.10) exist (and are unique) by [15, Theorem 4.18] for all  $\varepsilon > 0$  and  $\lambda \geq \lambda_0$ , where  $\lambda_0$  is a positive constant depending only on the coefficients of the Heston operator,  $A$  in (1.4) (see [15, Lemma 3.2]), chosen such that  $a_\lambda$  is coercive. We can now proceed to the

*Proof of Theorem 1.16.* Fix  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  as in the hypothesis of Theorem 1.16 and, with  $f_\lambda$  as in (6.12) with this choice of  $u$ , set

$$f_{\lambda, \varepsilon} := f_\lambda - \beta_\varepsilon(u_\varepsilon - \psi) \in L^2(\mathcal{O}, \mathfrak{w}). \quad (6.27)$$

Since  $f, \psi \in L^\infty(\mathcal{O})$  and  $u$  is a solution to the variational inequality (1.32), then  $u$  also solves

$$\begin{aligned} a_\lambda(u, v - u) & \geq (f_\lambda, v - u)_{L^2(\mathcal{O}, \mathfrak{w})} \quad \text{and} \quad u \geq \psi \text{ a.e. on } \mathcal{O}, \\ \forall v & \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}) \text{ with } v \geq \psi \text{ a.e. on } \mathcal{O}. \end{aligned}$$

We may assume that  $\lambda + r > 0$ , without loss of generality, and so the weak maximum principle for  $a_\lambda$  in [24, Proposition 7.9 & Theorem 8.15] implies that

$$\|u\|_{L^\infty(\mathcal{O})} < \frac{1}{\lambda + r} \|f\|_{L^\infty(\mathcal{O})} \vee \|\psi\|_{L^\infty(\mathcal{O})}, \quad (6.28)$$

where  $x \vee y := \max\{x, y\}$ , for all  $x, y \in \mathbb{R}$ . But

$$\|f_{\lambda,\varepsilon}\|_{L^s(\mathcal{O})} \leq \text{vol}^{1/s}(\mathcal{O}, \mathfrak{w}) \|f_{\lambda,\varepsilon}\|_{L^\infty(\mathcal{O})}, \quad \forall \varepsilon > 0,$$

where we take  $s > 2n \vee (n + \beta)$  and the bound (6.13) for  $\beta_\varepsilon(u_\varepsilon - \psi)$ , and the uniform bound (6.28) for  $u$  on  $\mathcal{O}$  imply that

$$\|f_{\lambda,\varepsilon}\|_{L^\infty(\mathcal{O})} \leq \|f\|_{L^\infty(\mathcal{O})} + \lambda(1 + \text{height}(\mathcal{O}))\|u\|_{L^\infty(\mathcal{O})} + 2 \operatorname{ess\,sup}_{\mathcal{O}}(A\psi - f)^+, \quad \forall \varepsilon \in (0, \varepsilon_0],$$

where  $\varepsilon_0 > 0$  is as in Lemma 6.3. Hence,  $f_{\lambda,\varepsilon}$  in (6.27) obeys the hypotheses of Corollary 1.12 and so, by application to the solution  $u_\varepsilon \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  to (6.10), that is

$$a_\lambda(u_\varepsilon, v) = (f_{\lambda,\varepsilon}, v)_{L^2(\mathcal{O}, \mathfrak{w})}, \quad \forall v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}),$$

we see that  $u_\varepsilon \in C_s^{\alpha_1}(\bar{\mathcal{O}}_{\delta_1})$ , for all  $\varepsilon \in (0, \varepsilon_0]$ , with  $\delta_1$  depending only on the height of the domain  $\mathcal{O}$ , and Hölder exponent  $\alpha_1 \in (0, 1)$  and Hölder norm  $C_1 > 0$  with the dependencies stated in Corollary 1.12, but independent of  $\varepsilon \in (0, \varepsilon_0]$ . Therefore, Corollary 1.12 implies that  $\{u_\varepsilon\}_{\varepsilon > 0} \subset C_s^{\alpha_1}(\bar{\mathcal{O}}_{\delta_1})$  and

$$\|u_\varepsilon\|_{C_s^{\alpha_1}(\bar{\mathcal{O}}_{\delta_1})} \leq C_1, \quad \forall \varepsilon \in (0, \varepsilon_0].$$

By the Arzelà-Ascoli Theorem, we can find a subsequence which converges uniformly on compact subsets of  $\bar{\mathcal{O}}_{\delta_1}$  to a function  $u_0 \in C_s^{\alpha_1}(\bar{\mathcal{O}}_{\delta_1})$ . But [15, Theorem 6.2] and the choice (6.12) of  $f_\lambda = f + \lambda(1 + y)u$  imply that  $u_\varepsilon \rightarrow u$  strongly in  $L^2(\mathcal{O}, \mathfrak{w})$  (in fact,  $H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ ) as  $\varepsilon \downarrow 0$  and thus, after passing to a subsequence,  $u_\varepsilon \rightarrow u$  pointwise a.e. on  $\mathcal{O}$  as  $\varepsilon \downarrow 0$ . Therefore, by choosing a diagonal subsequence, we obtain  $u = u_0$  a.e. on  $\mathcal{O}_{\delta_1}$ , and the result follows.  $\square$

## 7. HARNACK INEQUALITY

In this section, we prove Theorem 1.18, that is, the Harnack inequality for solutions  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  to the variational equation (1.18). The key differences from the proof of the classical Harnack inequality for weak solutions to non-degenerate elliptic equations [28, Theorem 8.20] are essentially those which we already outlined in §5 and the proof follows the same pattern as that of Theorem 1.11. Therefore, we only point out the major steps in the proof of Theorem 1.18, as the details were explained in the preceding sections. We now proceed to the

*Proof of Theorem 1.18.* For clarity, we split the proof into principal steps.

**Step 1** (Energy estimates). Let  $\eta \in C_0^1(\bar{\mathbb{H}})$  be a non-negative cutoff function with support in  $\bar{B}_{4R}(z_0)$ . Let  $\varepsilon > 0$  and

$$w = u + \varepsilon. \quad (7.1)$$

We consider  $\alpha \in \mathbb{R}$ ,  $\alpha \neq -1$ . We set  $H(w) = w^{(\alpha+1)/2}$  and

$$v = \eta^2 w^\alpha. \quad (7.2)$$

Then,  $v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  is a valid test function in (1.14) by Lemma A.5. By applying the same arguments as in the proofs of Theorem 1.7 and Theorem 1.11, we obtain the following analogous



energy estimate to (4.8) and (5.12), respectively

$$\begin{aligned} & \left( \int |\eta H(w)|^p y^{\beta-1} dx dy \right)^{1/p} \\ & \leq (C|1 + \alpha|)^{1/p} \|\sqrt{y} \nabla \eta\|_{L^\infty(\mathbb{H})}^{2/p} \left( \int_{\text{supp } \eta} |H(w)|^2 y^{\beta-1} dx dy \right)^{1/p}, \end{aligned} \quad (7.3)$$

where  $C$  is independent of  $\varepsilon$ , and depends only on the coefficients of the Heston operator,  $n$  and  $\bar{R}$ .

**Step 2** (Moser iteration). By applying Moser iteration as described in the proofs of Theorem 1.7, for  $\alpha > 0$ , and of Theorem 1.11, for  $\alpha < 0$ , we obtain

$$\begin{aligned} \sup_{B_R(z_0)} w & \leq C \left( \frac{1}{|B_{2R}(z_0)|^{\beta-1}} \int_{B_{2R}(z_0)} w^2 y^{\beta-1} dx dy \right)^{1/2}, \\ \inf_{B_R(z_0)} w & \geq C^{-1} \left( \frac{1}{|B_{2R}(z_0)|^{\beta-1}} \int_{B_{2R}(z_0)} w^{-2} y^{\beta-1} dx dy \right)^{-1/2}, \end{aligned} \quad (7.4)$$

where  $C$  satisfies the same dependencies as the constant in (7.3).

**Step 3** (Application of Theorem 3.4). In this step, we verify that  $w$  satisfies the requirements of the abstract John-Nirenberg inequality (Theorem 3.1) with  $\theta_0 = \theta_1 = 2$  and  $S_r = \mathbb{B}_{(2+r)R}(z_0)$ , for all  $0 \leq r \leq 1$ . From the hypotheses, we have that  $0 < 4R < \text{dist}(z_0, \Gamma_1)$ , and so  $S_r = \mathbb{B}_{(2+r)R}(z_0) = B_{(2+r)R}(z_0)$ , for all  $0 \leq r \leq 1$ , by (2.3) and (1.20). By Proposition 3.2, we see that  $w$  satisfies condition (3.2) of Theorem 3.1. Therefore, it remains to verify condition (3.3), which follows in precisely the same way as in the proof of Theorem 1.11.

**Step 4** (Proof of (1.37)). Because  $w$  satisfies the conditions of Theorem 3.1 by the preceding step, there is a positive constant  $C$ , independent of  $\varepsilon$ , such that

$$\begin{aligned} & \left( \frac{1}{|B_{2R}(z_0)|^{\beta-1}} \int_{B_{2R}(z_0)} w^2 y^{\beta-1} dx dy \right)^{1/2} \\ & \leq C \left( \frac{1}{|B_{2R}(z_0)|^{\beta-1}} \int_{B_{2R}(z_0)} w^{-2} y^{\beta-1} dx dy \right)^{-1/2}. \end{aligned} \quad (7.5)$$

Thus, combining inequalities (7.4) and (7.5) and recalling that  $w = u + \varepsilon$ , we obtain

$$\sup_{B_R(z_0)} (u + \varepsilon) \leq C \inf_{B_R(z_0)} (u + \varepsilon),$$

for all  $\varepsilon > 0$ . Taking the limit as  $\varepsilon \downarrow 0$ , we obtain the desired Harnack inequality (1.37).

This completes the proof.  $\square$

**Remark 7.1** (Hölder continuity and the Harnack inequality). We notice that the Hölder continuity of solutions, Theorem 1.11, does not follow from our Harnack inequality, Theorem 1.18, because in the latter theorem we assume that  $f = 0$  on  $B_{\bar{R}}(z_0)$ , while in Theorem 1.11 we allow for more general source functions,  $f$ . Moreover, our version of the Harnack inequality holds for points in  $\Gamma_0$ , and cannot be extended to points in  $\bar{\Gamma}_0 \setminus \Gamma_0$ . In order to obtain Theorem 1.11, we need to use a version of the *weak Harnack inequality* ([28, Theorem 8.18]), which is already embedded in our proof of Theorem 1.11 in estimate (5.32).

## APPENDIX A. AUXILIARY RESULTS

In this section we collect the technical justifications of a few assertions employed in the body of the article.

**A.1. A domain  $\mathcal{O}$  which does not satisfy condition (4.1).** Next, we give an example of a domain  $\mathcal{O}$  which does not satisfy condition (4.1).

**Example A.1.** This construction is in the spirit of [33, Example 4.2.17] (Lebesgue's thorn). Let  $z_0 = (0, 0)$ ,  $R_N = 1/N$  and  $a_N = N^{-2/\beta}$ . We set

$$\begin{aligned} C_N &= \{(x, y) \in \mathbb{B}_{R_N}(z_0) : 0 < y < a_N x\}, \\ C'_N &= \{(x, y) \in \mathbb{B}_{R_{N+1}}(z_0) : 0 < y < a_N x\}, \end{aligned}$$

and define  $\mathcal{O}$  by

$$\mathcal{O} = \bigcup_{N=1}^{\infty} C_N \setminus C'_N.$$

From Lemma 2.4, there exist positive constants  $c_1 < c_2$ , independent of  $R$  and  $N$ , such that

$$C_N \setminus C'_N \subseteq \{(x, y) \in \mathbb{H} : c_1 R_{N+1}^2 < x < c_2 R_N^2, 0 < y < a_N x\},$$

which give us

$$\begin{aligned} |C_N \setminus C'_N|_{\beta-1} &\leq \int_{c_1 R_{N+1}^2}^{c_2 R_N^2} \int_0^{a_N x} y^{\beta-1} dy dx \\ &= a_N^\beta \frac{c_2^{\beta+1} - c_1^{\beta+1}}{\beta(\beta+1)} (R_N^{2(1+\beta)} - R_{N+1}^{2(1+\beta)}) \\ &\leq \frac{C}{N^2} R_N^{2(1+\beta)} \\ &\leq \frac{C}{N^2} |\mathbb{B}_{R_N}(z_0)|_{\beta-1}, \quad (\text{by Lemma 2.4.}) \end{aligned}$$

Recall  $B_{R_N}(z_0) = \mathcal{O} \cap \mathbb{B}_{R_N}(z_0)$ . Then, we obtain

$$|B_{R_N}(z_0)|_{\beta-1} = \sum_{k=N}^{\infty} |C_k \setminus C'_k|_{\beta-1} \leq C |\mathbb{B}_{R_N}(z_0)|_{\beta-1} \sum_{k=N}^{\infty} \frac{1}{k^2},$$

which implies

$$\frac{|B_{R_N}(z_0)|_{\beta-1}}{|\mathbb{B}_{R_N}(z_0)|_{\beta-1}} \rightarrow 0, \quad \text{as } N \rightarrow \infty,$$

and so, we obtain a contradiction with the left hand side of (4.1).

**A.2. An extension lemma.** First, we give the proof of Lemma 2.7. As in §2, we work under the assumptions stated in Remark 2.8.

*Proof of Lemma 2.7.* By [15, Corollary A.14], it is enough to prove the existence of an extension operator for functions  $u \in C^1(\mathbb{B}_R(z_0))$ . Fix a point  $z'_0 = (x'_0, y'_0) \in \mathbb{B}_R(z_0)$ , say  $z'_0 = (R^2/100, R^2/100)$ . We consider two different cases depending on whether  $0 < y \leq y'_0$  or  $y > y'_0$ .

First, we consider the points  $z = (x, y) \in D \setminus \mathbb{B}_R(z_0)$  such that  $0 < y \leq y'_0$ . Let  $z' = (x', y)$  be the intersection of  $\partial \mathbb{B}_R(z_0)$  with the horizontal segment connecting  $z$  and  $(x'_0, y)$ . Then, we define  $Eu(z)$  by reflection (with respect to the point  $z'$  in the hyperplane at level  $y$ )

$$Eu(z) := u \left( x'_0 + \frac{|x' - x'_0|}{|x - x'_0|^2} (x - x'_0), y \right).$$

Next, we consider the case of points  $z = (x, y) \in D \setminus \mathbb{B}_R(z_0)$  such that  $y > y'_0$ . Let  $z' = (x', y')$  be the intersection of  $\partial \mathbb{B}_R(z_0)$  with the segment connecting  $z$  and  $z'_0$ . Then, we define  $Eu(z)$  by reflection

$$Eu(z) := u \left( z'_0 + \frac{|z' - z'_0|}{|z - z'_0|^2} (z - z'_0) \right).$$

It is clear that  $Eu$  is a continuous extension of  $u$  from  $\mathbb{B}_R(z_0)$  to  $D$ . Because  $\partial \mathbb{B}_R(z_0)$  is a piecewise smooth curve,  $Eu$  has well-defined weak derivatives in  $D$ . Next, we show that (2.10) holds. For this purpose, we denote by

$$\begin{aligned} D_1 &:= (D \setminus \mathbb{B}_R(z_0)) \cap \{y < y'_0\}, \\ D_2 &:= (D \setminus \mathbb{B}_R(z_0)) \cap \{y \geq y'_0\}. \end{aligned}$$

To prove (2.10), it is enough to show there is a positive constant  $C$ , depending on  $R$  and  $D$ , such that

$$\begin{aligned} \int_{D_1} |Eu(x, y)|^2 y^{\beta-1} dx dy &\leq C \int_{\mathbb{B}_R(z_0)} |u(x, y)|^2 y^{\beta-1} dx dy, \\ \int_{D_1} |\nabla Eu(x, y)|^2 y^\beta dx dy &\leq C \int_{\mathbb{B}_R(z_0)} |\nabla u(x, y)|^2 y^\beta dx dy, \\ \int_{D_2} |Eu(x, y)|^2 y^{\beta-1} dx dy &\leq C \int_{\mathbb{B}_R(z_0)} |u(x, y)|^2 y^{\beta-1} dx dy, \\ \int_{D_2} |\nabla Eu(x, y)|^2 y^\beta dx dy &\leq C \int_{\mathbb{B}_R(z_0)} |\nabla u(x, y)|^2 y^\beta dx dy, \end{aligned} \tag{A.1}$$

We begin by evaluating the integrals over  $D_1$  in (A.1) and we show that

$$\begin{aligned} \int_{D_1^+} |Eu(x, y)|^2 y^{\beta-1} dx dy &\leq C \int_{\mathbb{B}_R(z_0)} |u(x, y)|^2 y^{\beta-1} dx dy, \\ \int_{D_1^+} |\nabla Eu(x, y)|^2 y^\beta dx dy &\leq C \int_{\mathbb{B}_R(z_0)} |\nabla u(x, y)|^2 y^\beta dx dy, \end{aligned} \tag{A.2}$$

where  $D_1^+ := D_1 \cap \{x > 0\}$ . The analogous relation to (A.2) can be shown to hold on  $D_1^- := D_1 \cap \{x < 0\}$ , in a similar way.

Denote by

$$f(x, y) = x'_0 + \frac{|x' - x'_0|}{|x - x'_0|^2} (x - x'_0). \tag{A.3}$$

We notice that  $(f(x, y), y) \in \mathbb{B}_R(z_0)$ , for all  $(x, y) \in D_1$ , so  $Eu(x, y)$  is well-defined on  $D_1$ . The coordinate  $x' = x'(y)$  is determined by the condition  $d((y, x'), z_0) = R$ . Direct calculations give us

$$x'(y) = \left( \left( R^2 + R\sqrt{R^2 + 4y} \right)^2 / 4 - y^4 \right)^{1/2}.$$

We obtain, for all  $(x, y) \in D_1$ ,

$$\begin{aligned} f_x(x, y) &= -\frac{x' - x'_0}{(x - x'_0)^2}, \\ f_y(x, y) &= \frac{x'_y(y)}{x - x'_0}. \end{aligned}$$

We can find a positive constant  $C_1$ , depending only on  $R$ , such that

$$x - x'_0 \geq x' - x'_0 \geq C_1, \quad \forall (x, y) \in D_1^+,$$

and there is a positive constant  $C_2$ , depending on  $R$  and  $D$ , such that

$$|f_x(x, y)|, |f_x(x, y)|^{-1}, |f_y(x, y)| \leq C_2. \quad (\text{A.4})$$

Using the change of variable  $w = f(x, y)$  in (A.2), we obtain

$$\begin{aligned} \int_{D_1^+} |Eu(x, y)|^2 y^{\beta-1} dx dy &\leq \int_{\mathbb{B}_R(z_0)} |u(w, y)|^2 y^{\beta-1} |f_x(x, y)|^{-1} dw dy \\ &\leq C_2 \int_{\mathbb{B}_R(z_0)} |u(x, y)|^2 y^{\beta-1} dx dy, \quad (\text{by (A.4).}) \end{aligned} \quad (\text{A.5})$$

Using

$$\begin{aligned} \partial_x Eu(x, y) &= u_x(f(x, y), y) f_x(x, y), \\ \partial_y Eu(x, y) &= u_x(f(x, y), y) f_y(x, y) + u_y(f(x, y), y), \end{aligned}$$

the change of variable  $w = f(x, y)$  and the upper bound (A.4), we obtain for a positive constant  $C_3$ , depending on  $R$  and  $D$ ,

$$\int_{D_1^+} |\nabla Eu(x, y)|^2 y^\beta dx dy \leq C \int_{\mathbb{B}_R(z_0)} |\nabla u(w, y)|^2 (|f_x(x, y)|^2 + |f_y(x, y)|^2) |f_x(x, y)|^{-1} y^\beta dw dy,$$

and thus

$$\int_{D_1^+} |\nabla Eu(x, y)|^2 y^\beta dx dy \leq C_3 \int_{\mathbb{B}_R(z_0)} |\nabla u(x, y)|^2 y^\beta dx dy. \quad (\text{A.6})$$

Therefore, (A.5) and (A.6) give us (A.2).

Next, we consider the last two integrals in (A.1). Notice that on  $D_2$  we have  $y \geq y'_0 > 0$  and so it is enough to show

$$\begin{aligned} \int_{D_2} |Eu(x, y)|^2 dx dy &\leq C_4 \int_{\mathbb{B}_R(z_0)} |u(x, y)|^2 dx dy, \\ \int_{D_2} |\nabla Eu(x, y)|^2 dx dy &\leq C_4 \int_{\mathbb{B}_R(z_0)} |\nabla u(x, y)|^2 dx dy, \end{aligned} \quad (\text{A.7})$$

for some positive constant  $C_4$ , depending on  $R$  and  $D$ . For all  $(x, y) \in D_2$ , we denote

$$\varphi(x, y) \equiv (\varphi^1(x, y), \varphi^2(x, y)) := z'_0 + \frac{z' - z'_0}{|z - z'_0|^2} (z - z'_0).$$

Hence, we can find a positive constant  $C_5$ , depending on  $R$  and  $D$ , such that for all  $(x, y) \in D_2$ ,

$$\begin{aligned} \det |\nabla \varphi(x, y)|^{-1} &\leq C_5, \\ |\nabla \varphi(x, y)| &\leq C_5. \end{aligned} \quad (\text{A.8})$$

We notice that  $\varphi(x, y) \in \mathbb{B}_R(z_0)$ , for all  $(x, y) \in D_2$ . Therefore, using the change of variable  $w = \varphi(x, y)$ , we obtain

$$\begin{aligned} \int_{D_2} |Eu(x, y)|^2 dx dy &\leq \int_{\mathbb{B}_R(z_0)} |u(w)|^2 \det|\nabla\varphi(x, y)|^{-1} dw \\ &\leq C_5 \int_{\mathbb{B}_R(z_0)} |u(x, y)|^2 dx dy \quad (\text{by (A.8)}). \end{aligned} \quad (\text{A.9})$$

Using

$$\begin{aligned} \partial_x Eu(x, y) &= u_x(x, y)\varphi_x^1(x, y) + u_y(x, y)\varphi_x^2(x, y), \\ \partial_y Eu(x, y) &= u_x(x, y)\varphi_y^1(x, y) + u_y(x, y)\varphi_y^2(x, y), \end{aligned}$$

we obtain

$$\begin{aligned} \int_{D_2} |\nabla Eu(x, y)|^2 dx dy &\leq C \int_{\mathbb{B}_R(z_0)} |\nabla u(w)|^2 |\nabla\varphi(x, y)|^2 \det|\nabla\varphi(x, y)|^{-1} dx dy \\ &\leq CC_5 \int_{\mathbb{B}_R(z_0)} |\nabla u(x, y)|^2 dx dy, \quad (\text{by (A.8).}) \end{aligned} \quad (\text{A.10})$$

From (A.9) and (A.10), we obtain (A.7). This concludes the proof of Lemma 2.7.  $\square$

**A.3. Test functions.** In this subsection, we verify that the test functions used in the proofs of our main results are indeed in the space  $H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ . We start with the test function (4.11) used in the proof of Theorem 1.7.

**Lemma A.2.** *The function  $v$  given by (4.11) is in  $H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ .*

*Proof.* We only show that  $v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  defined by (4.11) with  $w = u^+ + A$ . The proof for the choice  $w = u^- + A$  follows similarly. We fix  $k \in \mathbb{N}$  and we consider the definitions of  $H_k$  and  $G_k$  given by (4.9) and (4.10), respectively.

Since  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ , we have  $u^+ \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  by [15, Lemma A.34]. Let  $\{u_i\}_{i \in \mathbb{N}}$  be a sequence of functions in  $C_0^1(\mathcal{O} \cup \Gamma_0)$  converging to  $u^+$  in  $H^1(\mathcal{O}, \mathfrak{w})$ . We extract a subsequence, for which we use the same notation as for the original sequence, such that

$$u_i \rightarrow u^+, \quad \text{a.e. on } \mathcal{O}. \quad (\text{A.11})$$

Let  $w_i := u_i + A$  and  $v_i := \eta G_k(w_i)$ , where  $\eta$  has support in  $\bar{\mathbb{B}}_{2R}(z_0)$  as in the proof of Theorem 1.7. Our goal is to show that  $v_i \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  converge to  $v$  in  $H^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ , from where the assertion of the lemma follows.

We notice that each  $v_i \in C(\bar{\mathcal{O}})$ . Because  $u_i = 0$  along  $\Gamma_1$  by construction, we have

$$w_i = A, \quad \text{along } \Gamma_1, \quad (\text{A.12})$$

and so, we also have by (4.9) and (4.10),

$$v_i = 0, \quad \text{along } \Gamma_1.$$

Since  $\eta$  has support in  $\bar{\mathbb{B}}_{2R}(z_0)$ , it follows that

$$v_i \in C_0(\mathcal{O} \cup \Gamma_0). \quad (\text{A.13})$$

Using

$$|H'_k(t)| \leq \alpha k^{\alpha-1}, \quad (\text{A.14})$$

we obtain

$$\begin{aligned} |v_i - v| &\leq \left| \int_{w_i}^w |H'_k(t)| dt \right| \leq \alpha k^{\alpha-1} |w_i - w| \\ &= \alpha k^{\alpha-1} |u_i - u^+|. \end{aligned}$$

Since the last term converges to zero in  $L^2(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ , it follows that

$$v_i \rightarrow v, \quad \text{as } i \rightarrow \infty, \quad \text{in } L^2(\mathcal{O} \cup \Gamma_0, \mathfrak{w}). \quad (\text{A.15})$$

By direct calculation, we have

$$\begin{aligned} \nabla v_i &= 2\eta \nabla \eta G_k(w_i) + \eta^2 |H'_k(w_i)|^2 \nabla u_i, \\ \nabla v &= 2\eta \nabla \eta G_k(w) + \eta^2 |H'_k(w)|^2 \nabla u^+. \end{aligned}$$

By (A.12), (A.14) and using  $\nabla u_i \in C_0(\mathcal{O} \cup \Gamma_0)$ , we obtain

$$\nabla v_i \in C_0(\mathcal{O} \cup \Gamma_0). \quad (\text{A.16})$$

We have

$$|\nabla v_i - \nabla v| \leq 2\eta |\nabla \eta| |G'_k(w_i) - G'_k(w)| + |H'_k(w_i)^2 - H'_k(w)^2| |\nabla u^+| + |\nabla u_i - \nabla u^+| |H'_k(w_i)|^2.$$

Using (A.14), there is a positive constant depending on  $k, \alpha$  and  $\eta$ , such that

$$|\nabla v_i - \nabla v| \leq C |u_i - u^+| + |H'_k(w_i)^2 - H'_k(w)^2| |\nabla u^+|. \quad (\text{A.17})$$

By (A.11) and the boundedness of  $H'_k$  in (A.14), we notice that

$$\begin{aligned} |H'_k(w_i)^2 - H'_k(w)^2| |\nabla u^+| &\leq |\alpha k^{\alpha-1}|^2 |\nabla u^+| \\ |H'_k(w_i)^2 - H'_k(w)^2| |\nabla u^+| &\rightarrow 0, \quad \text{as } i \rightarrow \infty \quad \text{a.e.}, \end{aligned}$$

and so, using the Dominated Convergence theorem, we have

$$|H'_k(w_i)^2 - H'_k(w)^2| |\nabla u^+| \rightarrow 0, \quad \text{as } i \rightarrow \infty, \quad \text{in } L^2(\mathcal{O} \cup \Gamma_0, y\mathfrak{w}).$$

Then, we obtain by (A.17)

$$|\nabla v_i - \nabla v| \rightarrow 0, \quad \text{as } i \rightarrow \infty, \quad \text{in } L^2(\mathcal{O} \cup \Gamma_0, y\mathfrak{w}).$$

Combining the preceding inequality with (A.13), (A.15) and (A.16), we obtain the assertion of the lemma.  $\square$

Next, we verify that the test functions employed in the proofs of Theorems 1.11 and 1.18 belong to the space  $H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ . For this purpose, since  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ , we let  $\{u_i\}_{i \in \mathbb{N}}$  be a sequence of functions in  $C_0^1(\mathcal{O} \cup \Gamma_0)$  converging to  $u$  in  $H^1(\mathcal{O}, \mathfrak{w})$ . We extract a subsequence, for which we keep the same notation as for the original sequence, such that

$$u_i \rightarrow u, \quad \text{a.e. on } \mathcal{O}. \quad (\text{A.18})$$

We will use this construction in the following results of this subsection.

**Lemma A.3.** *The function  $v$  given by (5.10) is in  $H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ , for any  $\alpha \in \mathbb{R}$ .*

*Proof.* We outline the proof for the choice  $w = u - m_{4R} + k(R)$  in (5.7) in the definition of  $v$  in (5.10). The conclusion of the lemma for the choice  $w = M_{4R} - u + k(R)$  in (5.10) follows similarly. Let

$$\Omega_i := \{z \in B_{4R}(z_0) : -k/2 + m_{4R} \leq u_i \leq M_{4R} + k/2\},$$

and let  $\Omega_i^c$  be the complement of  $\Omega_i$  in  $B_{4R}(z_0)$ . By setting

$$\hat{u}_i := (u_i \wedge (-k/2 + m_{4R})) \vee (M_{4R} + k/2), \quad \forall i \in \mathbb{N},$$

we obtain

$$\begin{aligned} \int_{B_{4R}(z_0)} |u_i - u|^2 \mathfrak{w} \, dx \, dy &\geq \int_{\Omega_i^c} |u_i - u|^2 \mathfrak{w} \, dx \, dy \\ &\geq (k/2)^2 |\Omega_i^c|_{\mathfrak{w}}. \end{aligned}$$

Since the left hand side in the preceding inequality converges to zero, we obtain that

$$|\Omega_i^c|_{\mathfrak{w}} \rightarrow 0, \quad \text{as } i \rightarrow \infty. \quad (\text{A.19})$$

We let

$$w_i := \hat{u}_i - m_{4R} + k$$

Then,  $w_i$  satisfies on  $B_{4R}(z_0)$

$$k/2 \leq w_i \leq M_{4R} - m_{4R} + 3k/2. \quad (\text{A.20})$$

Now, we define

$$v_i := \eta^2 w_i^\alpha, \quad \forall i \in \mathbb{N},$$

where  $\alpha \in \mathbb{R}$  and  $\eta$  is a smooth, non-negative cutoff function with support in  $\bar{B}_{4R}(z_0)$ . By (5.1) and (A.20), we notice that  $v_i$  are well-defined functions and

$$v_i \in C_0(\mathcal{O} \cup \Gamma_0), \quad \forall i \in \mathbb{N}.$$

By (A.18) and (A.19), we obtain that  $v_i$  converges a.e. to  $v$ , and by (A.20), the sequence  $\{v_i\}_{i \in \mathbb{N}}$  is uniformly bounded. Thus, by the Dominated Convergence theorem we obtain that the sequence  $\{v_i\}_{i \in \mathbb{N}}$  converges to  $v$  in  $L^2(\mathcal{O}, \mathfrak{w})$ .

Next, we have

$$\begin{aligned} \nabla v_i &:= 2\eta \nabla \eta w_i^\alpha + \alpha \eta w_i^{\alpha-1} \nabla \hat{u}_i, \\ \nabla v &:= 2\eta \nabla \eta w^\alpha + \alpha \eta w^{\alpha-1} \nabla u. \end{aligned}$$

Since the support of  $\eta$  is included in  $\bar{B}_{4R}(z_0)$ , the same holds for  $\nabla v_i$ , for all  $i \in \mathbb{N}$ . We can evaluate  $\nabla v_i - \nabla v$  in the following way. There exists a positive constant  $C$ , depending only on  $\eta$  and  $\alpha$ , such that on  $B_{4R}(z_0)$

$$\begin{aligned} |\nabla v_i - \nabla v| &\leq C|w_i^\alpha - w^\alpha| + C|w_i^{\alpha-1} \nabla \hat{u}_i - w^{\alpha-1} \nabla u| \\ &\leq C|w_i^\alpha - w^\alpha| + C|w_i^{\alpha-1}| |\nabla \hat{u}_i - \nabla u| + C|w_i^{\alpha-1} - w^{\alpha-1}| |\nabla u|. \end{aligned} \quad (\text{A.21})$$

Recall that  $\{w_i\}_{i \in \mathbb{N}}$  converges a.e. to  $w$  on  $B_{4R}(z_0)$ . By (A.20), for any  $t \in \mathbb{R}$ , the sequence  $\{w_i^t\}_{i \in \mathbb{N}}$  is uniformly bounded, and so we have by the Dominated Convergence theorem, that  $|w_i^\alpha - w^\alpha|$  and  $|w_i^{\alpha-1} - w^{\alpha-1}| |\nabla u|$  converges to zero in  $L^2(B_{4R}(z_0), y\mathfrak{w})$ . Moreover, by (A.20), there is a positive constant  $C$ , such that on  $B_{4R}(z_0)$

$$|w_i^{\alpha-1}| |\nabla \hat{u}_i - \nabla u| \leq C |\nabla \hat{u}_i - \nabla u|.$$

Notice that

$$\begin{aligned} \int_{B_{4R}(z_0)} |\nabla \hat{u}_i - \nabla u|^2 y \mathfrak{w} \, dx \, dy &= \int_{\Omega_i} |\nabla u_i - \nabla u|^2 y \mathfrak{w} \, dx \, dy + \int_{B_{4R}(z_0) \setminus \Omega_i} |\nabla u|^2 y \mathfrak{w} \, dx \, dy \\ &\leq \int_{B_{4R}(z_0)} |\nabla u_i - \nabla u|^2 y \mathfrak{w} \, dx \, dy + \int_{B_{4R}(z_0)} \chi_{\Omega_i^c} |\nabla u|^2 y \mathfrak{w} \, dx \, dy. \end{aligned}$$

The first term in the preceding inequality converges to zero, because  $\{u_i\}_{i \in \mathbb{N}}$  converges to  $u$  in  $H^1(\mathcal{O}, \mathfrak{w})$ , and the second term converges to zero as well, by (A.19) and because  $\nabla u \in L^2(\mathcal{O}, y\mathfrak{w})$ . Therefore, we conclude by (A.21) that  $\nabla v_i$  converges to  $\nabla v$  in  $L^2(B_{4R}(z_0), y\mathfrak{w})$ , and so the conclusion of the lemma follows.  $\square$



**Lemma A.4.** *The function  $v^\pm$  given by (5.58) is in  $H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ , for all  $\alpha < 0$ .*

*Proof.* We recall the definition of the test functions  $v^\pm$  given by (5.58). For  $\alpha < 0$ , we let

$$v^\pm := \eta^2 [(w^\pm)^\alpha - (k + M_{4R}^\pm)^\alpha],$$

where  $\eta$  is a smooth function with support in  $\mathbb{B}_{4R}(z_0)$  and  $w^\pm$  is defined by

$$w^\pm(z) := k + \begin{cases} -u^\pm(z) + M_{4R}^\pm, & z \in \mathbb{B}_{4R}(z_0) \cap B_{4R}(z_0), \\ +M_{4R}^\pm, & z \in \mathbb{B}_{4R}(z_0) \setminus B_{4R}(z_0), \end{cases}$$

and  $M_{4R}^\pm := \text{ess sup}_{B_{4R}(z_0)} u^\pm$  by (5.55), and  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ . Because  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ , it follows that  $u^+$  and  $u^-$  are in  $H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  by [15, Lemma A.33] and, by definition,  $u^+$  and  $u^-$  are non-negative. The functions  $v^+$  and  $w^+$  depend only on  $u^+$ , while  $v^-$  and  $w^-$  depend only on  $u^-$ . If we show that  $v^+$  and  $v^-$  belong to  $H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ , then it will follow immediately that  $v = v^+ - v^-$  belongs to  $H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ . Since the proofs for  $v^+$  and  $v^-$  will be identical, it suffices to consider  $v^+$  alone and, furthermore, since  $v^+$ ,  $w^+$ , and  $M_r^+$  are defined in terms of  $u^+$ , we may drop the  $+$  superscript to simplify notation and assume without loss of generality that  $u \geq 0$  a.e. on  $\mathcal{O}$  and  $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$  and write  $v$ ,  $w$ , and  $M_r$  instead of  $v^+$ ,  $w^+$ , and  $M_r^+$  for the remainder of the proof. From Theorem 1.7, we know that  $u$  is bounded on  $B_{4R}(z_0)$  and we have by (1.22)

$$0 \leq u \leq M_{4R}, \quad \text{a.e. on } B_{4R}(z_0),$$

which implies

$$k \leq w \leq k + M_{4R}, \quad \text{a.e. on } B_{4R}(z_0). \quad (\text{A.22})$$

Recall that we may assume without loss of generality that  $M_{4R} \neq 0$  and  $k \neq 0$ , by (5.57) and (5.9), respectively. Notice that if  $M_{4R} = 0$ , then  $u \equiv 0$  on  $B_{4R}(z_0)$  and also  $v \equiv 0$  on  $\mathcal{O}$ , so the conclusion of the lemma holds trivially. Let  $\{u_i : i \in \mathbb{N}\} \subset C_0^\infty(\mathcal{O} \cup \Gamma_0)$  be a sequence of smooth functions with compact support in  $\mathcal{O} \cup \Gamma_0$  converging to  $u$  in  $H^1(\mathcal{O}, \mathfrak{w})$ . By the boundedness of  $u$ , and positivity of  $k$  and  $M_{4R}$ , we may assume without loss of generality that

$$-M_{4R}/2 \leq u_i \leq M_{4R} + k/2, \quad \forall i \in \mathbb{N}, \quad (\text{A.23})$$

and that we have the a.e. convergence on  $\mathcal{O}$

$$u_i \rightarrow u, \quad \text{as } i \rightarrow \infty. \quad (\text{A.24})$$

Let

$$w_i(z) := k + \begin{cases} -u_i(z) + M_{4R}, & z \in \mathbb{B}_{4R}(z_0) \cap B_{4R}(z_0), \\ +M_{4R}, & z \in \mathbb{B}_{4R}(z_0) \setminus B_{4R}(z_0), \end{cases} \quad \forall i \in \mathbb{N},$$

and set

$$v_i := \eta^2 (w_i^\alpha - (k + M_{4R})^\alpha), \quad \forall i \in \mathbb{N}.$$

Note that by the definition of the functions  $w_i$  and (A.23), we have

$$k + 3M_{4R}/2 \geq w_i \geq k/2, \quad \forall i \in \mathbb{N}, \quad (\text{A.25})$$

so that  $v_i$  is a well-defined function, for all  $\alpha < 0$  and  $i \in \mathbb{N}$ . Also,  $v_i \in C^\infty(\bar{\mathcal{O}})$ ,  $v_i = 0$  along  $\Gamma_1$ , and so  $v_i \in C_0^\infty(\mathcal{O} \cup \Gamma_0)$ , since the support of  $\eta$  is contained in  $\mathbb{B}_{4R}(z_0)$ . We want to prove that  $v_i$  converges to  $v$  in  $H^1(\mathcal{O}, \mathfrak{w})$  and hence that  $v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ . Notice that the definitions of the sequences  $\{v_i\}_{i \in \mathbb{N}}$  and  $\{w_i\}_{i \in \mathbb{N}}$  are motivated by the definition (5.58) of  $v$  and (5.56) of  $w$ , respectively.

By (A.24) and the positivity of  $w$  and  $\{w_i\}_{i \in \mathbb{N}}$ , we also have that, as  $i \rightarrow \infty$ ,

$$v_i \rightarrow v, \quad \text{a.e. on } B_{4R}(z_0), \quad (\text{A.26})$$

$$w_i^t \rightarrow w^t, \quad \text{a.e. on } B_{4R}(z_0), \quad \forall t \in \mathbb{R}, \quad (\text{A.27})$$

where  $t$  denotes any real power in this case. In addition, by (A.25) and (A.22) we can find a positive constant  $N_1$ , depending on  $\alpha$ , such that

$$\begin{aligned} \|v\|_{L^\infty(\mathcal{O})} &\leq N_1, \\ \|v_i\|_{L^\infty(\mathcal{O})} &\leq N_1, \quad \forall i \in \mathbb{N}, \end{aligned} \quad (\text{A.28})$$

and, for any  $t \in \mathbb{R}$ , we can find positive constants  $N_2$ , depending on  $t$ , such that

$$\begin{aligned} \|w^t\|_{L^\infty(B_{4R}(z_0))} &\leq N_2, \\ \|w_i^t\|_{L^\infty(B_{4R}(z_0))} &\leq N_2, \quad \forall i \in \mathbb{N}. \end{aligned} \quad (\text{A.29})$$

Therefore, using the Dominated Convergence theorem, (A.26) and (A.28), we obtain

$$v_i \rightarrow v, \quad \text{in } L^2(\mathcal{O}, \mathfrak{w}). \quad (\text{A.30})$$

It remains to establish that

$$\nabla v_i \rightarrow \nabla v, \quad \text{in } L^2(\mathcal{O}, y\mathfrak{w}), \quad \text{as } i \rightarrow \infty. \quad (\text{A.31})$$

By a direct calculation, we have

$$\nabla v = 2\eta \nabla \eta (w^\alpha - (k + M_{4R})^\alpha) + \alpha \eta^2 w^{\alpha-1} \nabla u, \quad (\text{A.32})$$

$$\nabla v_i = 2\eta \nabla \eta (w_i^\alpha - (k + M_{4R})^\alpha) + \alpha \eta^2 w_i^{\alpha-1} \nabla u_i, \quad \forall i \in \mathbb{N}. \quad (\text{A.33})$$

We denote by  $V^1$  and  $V^2$  the two terms appearing on the right-hand side of (A.32). Analogously, we denote by  $V_i^1$  and  $V_i^2$ ,  $i \in \mathbb{N}$ , the two terms on the right-hand side in (A.33). Next, we show that  $V_i^k$  converges in  $L^2(\mathcal{O}, y\mathfrak{w})$  to  $V^k$ , for  $k = 1, 2$ , which implies (A.31). By choosing  $t = \alpha$  in (A.27) and (A.29), we obtain using the Dominated Convergence theorem that  $V_i^1$  converges to  $V^1$  in  $L^2(\mathcal{O}, y\mathfrak{w})$ . Next, we have

$$|V_i^2 - V^2| \leq |\alpha| \eta^2 |w_i^{\alpha-1} - w^{\alpha-1}| |\nabla u| + |\alpha| \eta^2 |w_i|^{\alpha-1} |\nabla u_i - \nabla u|,$$

and, using (A.29) with  $t = \alpha - 1$ , for the second term on the right-hand side, we have

$$|V_i^2 - V^2| \leq |\alpha| \eta^2 |w_i^{\alpha-1} - w^{\alpha-1}| |\nabla u| + |\alpha| N_2 |\nabla u_i - \nabla u|, \quad \forall i \in \mathbb{N}.$$

Obviously, the second term in the preceding inequality converges to zero in  $L^2(\mathcal{O}, y\mathfrak{w})$ . The first term  $|w_i^{\alpha-1} - w^{\alpha-1}| |\nabla u|$  converges to zero a.e., by (A.27), and by (A.29), it satisfies the upper bound

$$\eta^2 |w_i^{\alpha-1} - w^{\alpha-1}| |\nabla u| \leq 2N_2 |\nabla u|, \quad \forall i \in \mathbb{N}.$$

Since  $\nabla u \in L^2(\mathcal{O}, y\mathfrak{w})$ , we may apply the Dominated Convergence theorem to conclude that

$$\eta^2 |w_i^{\alpha-1} - w^{\alpha-1}| |\nabla u| \rightarrow 0 \quad \text{in } L^2(\mathcal{O}, y\mathfrak{w}).$$

Therefore, we obtain that  $V_i^2$  converges in  $L^2(\mathcal{O}, y\mathfrak{w})$  to  $V^2$ , and so, (A.31) follows. Combining (A.30) and (A.31), we obtain that  $\{v_i\}_{i \in \mathbb{N}}$  is a sequence of functions in  $H_0^1(\mathcal{O} \cup \Gamma_0, y\mathfrak{w})$  converges to  $v$ , and so,  $v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ .  $\square$

Next, we show that the test function used in the proof of the Harnack inequality, Theorem 1.18, is indeed in  $H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ .

**Lemma A.5.** *The function  $v$  given by (7.2) is in  $H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ .*

*Proof.* The proof of the lemma follows similarly to the proof of Lemma A.3. Because of this, we only outline the main steps of the proof. Let

$$\Omega_i := \{z \in B_{4R}(z_0) : -\varepsilon/2 \leq u_i \leq M_{4R} + \varepsilon/2\}, \quad \forall i \in \mathbb{N}.$$

and

$$\hat{u}_i := (u_i \wedge (-\varepsilon/2)) \vee (M_{4R} + \varepsilon/2), \quad \forall i \in \mathbb{N}.$$

We let

$$w_i := \hat{u}_i + \varepsilon, \quad \forall i \in \mathbb{N}.$$

Then,  $w_i$  satisfies on  $B_{4R}(z_0)$

$$\varepsilon/2 \leq w_i \leq M_{4R} + 3\varepsilon/2, \quad \forall i \in \mathbb{N}.$$

Now, we define

$$v_i := \eta^2 w_i^\alpha, \quad \forall i \in \mathbb{N},$$

where  $\eta$  is a smooth, non-negative cutoff function with support in  $\bar{B}_{4R}(z_0)$ . Similarly to the proof of Lemma A.3, it follows that  $v_i \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ , for all  $i \in \mathbb{N}$ , and  $\{v_i\}_{i \in \mathbb{N}}$  converges to  $v$  in  $H^1(\mathcal{O}, \mathfrak{w})$ , and thus the conclusion of the lemma follows.  $\square$

**A.4. Weighted Sobolev norms and uniform bounds.** We have the following analogue of [1, Theorem 2.8], [28, Exercise 7.1].

**Lemma A.6** (Weighted Sobolev norms and uniform bounds). *For  $1 \leq p < \infty$  and  $u$  a measurable function on  $\mathcal{O}$  such that  $|u|^p \in L^1(\mathcal{O}, \mathfrak{w})$  for some  $p \in \mathbb{R}$ , define*

$$\Phi_p(u) := \left( \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} |u|^p \mathfrak{w} \, dx \, dy \right)^{1/p}.$$

Then

$$\lim_{p \rightarrow \infty} \Phi_p(u) = \operatorname{ess\,sup}_{\mathcal{O}} |u|, \tag{A.34}$$

$$\lim_{p \rightarrow -\infty} \Phi_p(u) = \operatorname{ess\,inf}_{\mathcal{O}} |u|. \tag{A.35}$$

*Proof.* For  $1 \leq p < q < \infty$ ,

$$\int_{\mathcal{O}} |u|^p \mathfrak{w} \, dx \, dy \leq \left( \int_{\mathcal{O}} |u|^q \mathfrak{w} \, dx \, dy \right)^{p/q} \left( \int_{\mathcal{O}} 1 \mathfrak{w} \, dx \, dy \right)^{1-p/q},$$

and thus

$$\Phi_p(u) \leq \Phi_q(u),$$

while for  $q = \infty$ ,

$$\int_{\mathcal{O}} |u|^p \mathfrak{w} \, dx \, dy \leq \left( \operatorname{ess\,sup}_{\mathcal{O}} |u| \right)^p \int_{\mathcal{O}} 1 \mathfrak{w} \, dx \, dy,$$

and thus

$$\Phi_p(u) \leq \operatorname{ess\,sup}_{\mathcal{O}} |u|.$$

Hence,

$$\lim_{p \rightarrow \infty} \Phi_p(u) \leq \operatorname{ess\,sup}_{\mathcal{O}} |u|.$$

On the other hand, for any  $\varepsilon > 0$ , there is a set  $B \subset \mathcal{O}$  of positive measure  $|B| = \int_B 1 \mathfrak{w} \, dx \, dy$  such that

$$|u(x)| \geq \operatorname{ess\,sup}_{\mathcal{O}} |u| - \varepsilon, \quad x \in B.$$

Hence,

$$\frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} |u|^p \mathfrak{w} \, dx \, dy \geq \frac{1}{|\mathcal{O}|} \int_B |u|^p \mathfrak{w} \, dx \, dy \geq \frac{|B|}{|\mathcal{O}|} \left( \operatorname{ess\,sup}_{\mathcal{O}} |u| - \varepsilon \right)^p,$$

so

$$\left( \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} |u|^p \mathfrak{w} \, dx \, dy \right)^{1/p} \geq \left( \frac{|B|}{|\mathcal{O}|} \right)^{1/p} \left( \operatorname{ess\,sup}_{\mathcal{O}} |u| - \varepsilon \right)$$

It follows that  $\Phi_p(u) \geq (|B|/|\mathcal{O}|)^{1/p} (\operatorname{ess\,sup}_{\mathcal{O}} |u| - \varepsilon)$ , and thus

$$\lim_{p \rightarrow \infty} \Phi_p(u) \geq \operatorname{ess\,sup}_{\mathcal{O}} |u|.$$

For the second assertion, we may assume without loss of generality that  $\operatorname{ess\,inf}_{\mathcal{O}} |u| > 0$  and so  $\operatorname{ess\,sup}_{\mathcal{O}} |u|^{-1} = (\operatorname{ess\,inf}_{\mathcal{O}} |u|)^{-1}$ . For  $1 \leq p < q < \infty$ ,

$$\int_{\mathcal{O}} |u|^{-p} \mathfrak{w} \, dx \, dy \leq \left( \int_{\mathcal{O}} |u|^{-q} \mathfrak{w} \, dx \, dy \right)^{p/q} \left( \int_{\mathcal{O}} 1 \mathfrak{w} \, dx \, dy \right)^{1-p/q},$$

so

$$\left( \int_{\mathcal{O}} |u|^{-p} \mathfrak{w} \, dx \, dy \right)^{-p} \geq \left( \int_{\mathcal{O}} |u|^{-q} \mathfrak{w} \, dx \, dy \right)^{-q}$$

and thus

$$\Phi_{-p}(u) \geq \Phi_{-q}(u),$$

while for  $q = -\infty$ ,

$$\int_{\mathcal{O}} |u|^{-p} \mathfrak{w} \, dx \, dy \leq \left( \operatorname{ess\,sup}_{\mathcal{O}} |u|^{-1} \right)^p \int_{\mathcal{O}} 1 \mathfrak{w} \, dx \, dy,$$

and thus

$$\Phi_{-p}(u) = \left( \int_{\mathcal{O}} |u|^{-p} \mathfrak{w} \, dx \, dy \right)^{-p} \geq \left( \operatorname{ess\,sup}_{\mathcal{O}} |u|^{-1} \right)^{-1} = \operatorname{ess\,inf}_{\mathcal{O}} |u|.$$

Hence,

$$\lim_{p \rightarrow \infty} \Phi_{-p}(u) \geq \operatorname{ess\,inf}_{\mathcal{O}} |u|.$$

On the other hand, for any  $\varepsilon > 0$ , there is a set  $B \subset \mathcal{O}$  of positive measure such that

$$|u(x)| \leq \operatorname{ess\,inf}_{\mathcal{O}} |u| + \varepsilon, \quad x \in B.$$

Hence,

$$\frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} |u|^{-p} \mathfrak{w} \, dx \, dy \geq \frac{1}{|\mathcal{O}|} \int_B |u|^{-p} \mathfrak{w} \, dx \, dy \geq \frac{|B|}{|\mathcal{O}|} \left( \operatorname{ess\,inf}_{\mathcal{O}} |u| + \varepsilon \right)^{-p}.$$

It follows that  $\Phi_{-p}(u) \leq (|B|/|\mathcal{O}|)^{-1/p} (\operatorname{ess\,inf}_{\mathcal{O}} |u| + \varepsilon)$ , and thus

$$\lim_{p \rightarrow \infty} \Phi_{-p}(u) \leq \operatorname{ess\,inf}_{\mathcal{O}} |u|.$$

This completes the proof.  $\square$

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